9 Sep 2024 Online Matching

Announcements.

① P. Set 1 due Fri, 9/13, 11:59 pm
with 48 hour grace period.

② Office hours
- Linda Tues 11:30–12:30 (RH 574)
- August Wed 10–11 (RH 400)
- me Wed 3:30–5 (Gates 317)
Linda subbing this week

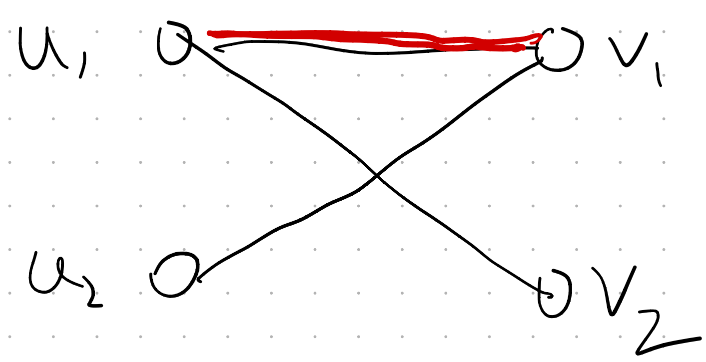
In the online matching problem there's a bipartite graph $G = (V, E)$, $V = L \cup R$, but:
- the vertices in L are present at time zero.
- the vertices in R are $v_1, v_2, \ldots, v_T$ and v_t arrives at time $t \in \{1, \ldots, T\}$.
- Vertices in R must be matched upon arrival, or else remain unmatched forever.
- The neighbors of v_t become known when it arrives, not before.
- Once a match (u, v) is made it can't be broken off.

How to evaluate algorithms?

Goal. Make as many matches as possible.

i.e., max-cardinality matching.

Not always possible...



For any online algorithm there are some input sequences where it makes only one match, though in hindsight two were possible.

There is a simple algorithm that makes at least half as many matches as the max matching, on every possible input sequence.

"GREEDY is 2-competitive."

The algorithm is GREEDY. When v_t arrives if it has an unmatched neighbor, match it to any such neighbor.

Let M^* and M_G denote a max matching and a greedy matching, respectively.

$$L^* = \{\text{left endpoints of edges in } M^*\}$$
$$L_G = \{ \text{-- -- -- -- -- } \ldots M_G\}.$$
$$R^* = \{\text{right endpoints of edges in } M^*\}$$
$$R_G = \{ \text{-- -- -- -- -- } \ldots M_G\}.$$

For each edge $v \in R^* \setminus R_G$ there is an edge $(u,v) \in M^*$ with $u \in L_G$.

This gives an injection from $R^* \setminus R_G \to L_G$.

$$|M^*| = |R^*| = |R^* \cap R_G| + |R^* \setminus R_G|$$
$$\le |R_G| + |L_G|$$
$$= 2 \cdot |M_G|$$

For deterministic algorithms:

- any greedy algorithm is 2-competitive
- no algorithm is better than 2-competitive.

For randomized algorithms, the goal is:

maximize <u>expected</u> number of edges in the matching.

A randomized algorithm that outputs a (random) matching M_{RAND}, is α-competitive if it satisfies

$$\mathbb{E}\left[M_{RAND} \right] \geq \frac{1}{\alpha} \cdot |M^*|$$

on every input sequence with M^* denoting max-matching.

RANKING (Karp-Vazirani-Vazirani)

GREEDY with random consistent tie-breaking.

At time ϕ, sample a uniformly random permutation of L.

Each time a vertex v_t arrives with 1 or more free neighbors, match to the free neighbor that occurs earliest in this rand perm.