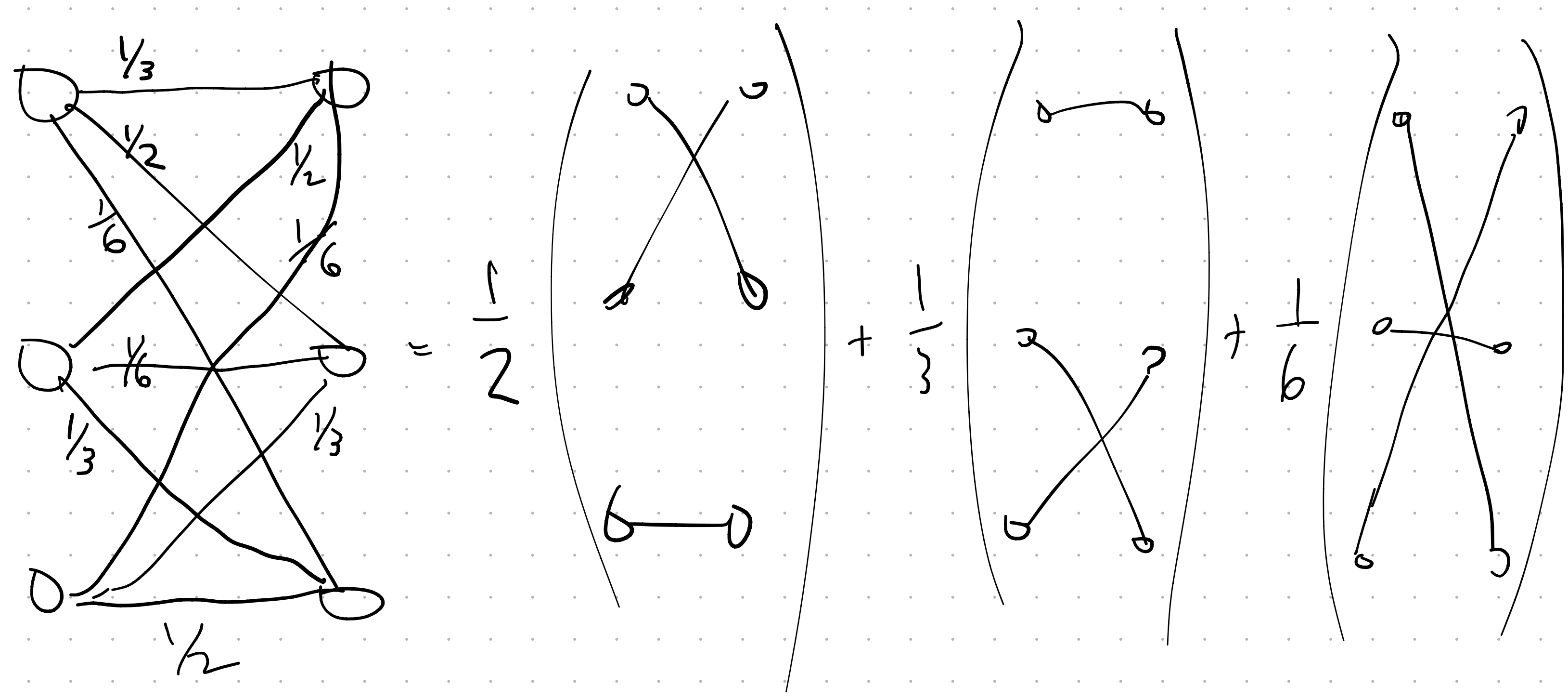
6 Sept 2024

LP Relaxation of Bipartite Matching

A <u>fractional matching</u> in a bipartite graph is a vector $x = (x_{uv})_{\{u,v\} \in E}$ that satisfies

$$\forall u \in L \quad \sum_v x_{uv} \leq 1$$
$$\forall v \in R \quad \sum_u x_{uv} \leq 1$$
$$\forall \{u,v\} \in E \quad x_{uv} \geq 0.$$

Example of a fractional matching...



A fractional <u>perfect</u> matching satisfies $\sum_v x_{uv} = 1 \ \forall u$ and $\sum_u x_{uv} = 1 \ \forall v$.

The cost of a fractional matching is

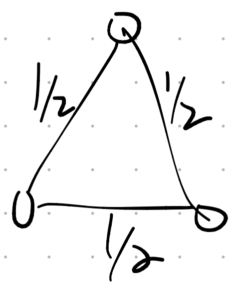
$$\text{cost}(\vec{x}) = \sum_{\{u,v\} \in E} c_{uv} x_{uv}$$

This problem "relaxes" min-cost bipartite perfect matching by enlarging the set of potential solutions.

E.g. fractional matching in non-bipartite graph could be defined by

$$\forall u \quad \sum_{v} x_{\{u,v\}} \leq 1$$

$$\vec{x} \geq 0$$

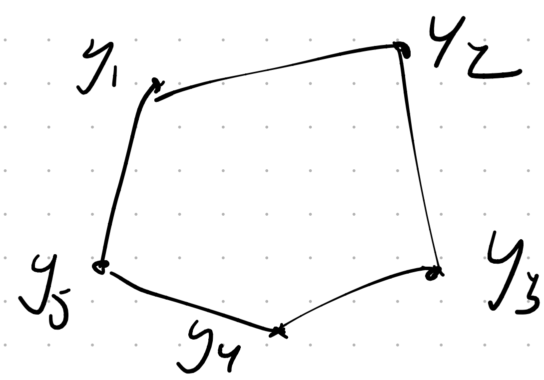


A fractional matching that is "larger" than any actual matching.

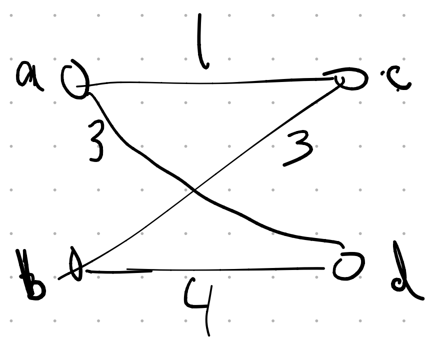
<u>Thm.</u> (Birkhoff – von Neumann) In any bipartite graph every fractional perfect matching is a <u>convex</u> <u>combination</u> of integer-valued perfect matchings.

x is a convex combination of $y_1, \ldots, y_M$ means $\exists$ weights $w_1, \ldots, w_M \geq 0$ $\sum_{j=1}^{M} w_j = 1$ s.t. $x = w_1 y_1 + w_2 y_2 + \cdots + w_M y_M$.

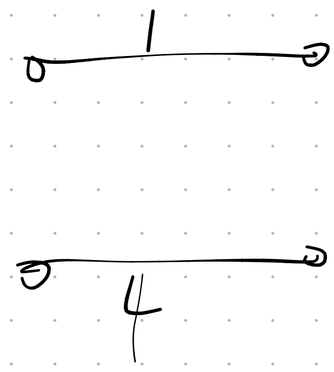
"x is a weighted average of $y_1, \ldots, y_m$"



<u>Corollary</u> cost (min-cost fractional PM) = cost (min-cost perfect matching).



Claim:



is min cost perfect matching.

For any fractional perfect matching x,

$$\text{cost}(x) = x_{ac} + 3x_{ad} + 3x_{bc} + 4x_{bd}$$

$$x_{ac} + x_{ad} = 1 \qquad -x_{ac} - x_{ad} = -1$$

$$x_{bc} + x_{bd} = 1$$

$$x_{ac} + x_{bc} = 1 \qquad 2x_{ac} + 2x_{bc} = 2$$

$$x_{ad} + x_{bd} = 1 \qquad 4x_{ad} + 4x_{bd} = 4$$

$$x_{ac} + 3x_{ad} + 2x_{bc} + 4x_{bd} = 5$$

Lower bound on cost of any matching

To bound from below the cost of any fractional PM, we can choose coefficients y_u $(u \in V)$, that will be used to scale the degree constraints.

These can be positive, negative, or zero.

Scale constraints by y_u's and sum up.

$$\sum_{u\in L} y_u \left(\sum_v x_{uv}\right) + \sum_{v\in R} y_v \left(\sum_u x_{uv}\right) = \sum_{u\in L} y_u + \sum_{v\in R} y_v$$

$$\sum_{\{u,v\}\in E} (y_u + y_v) x_{uv} = \sum_{u\in L} y_u + \sum_{v\in R} y_v$$

Always satisfied for any f.p.m. x and any vector y of scaling coeffs.

For this equation to be useful we want

$$y_u + y_v \leq c_{uv} \qquad \forall \{u,v\}\in E.$$

Vectors y satisfying these relations are called "dual solutions" for min-cost fractional perfect matching.

For the dual solution to "certify optimality" of a fractional perfect matching, x, we need

$$\text{cost}(x) = \text{dual-cost}(y) = \sum_{u\in L} y_u + \sum_{v\in R} y_v$$