4 Sep 2024

Finish Hopcroft – Karp

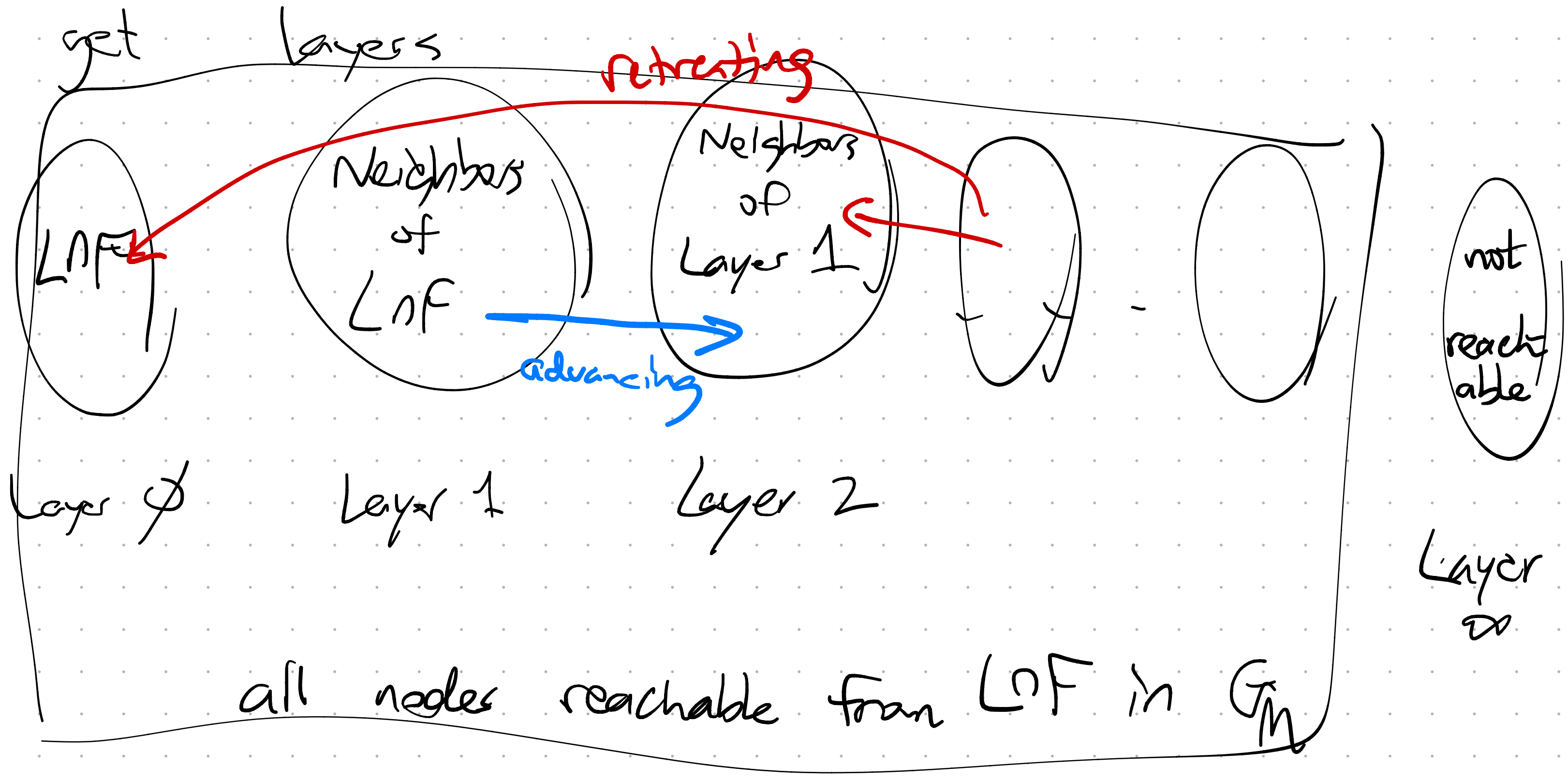
Talk about LP relaxation of bipartite matching

<u>Reminder.</u> G bipartite graph

M matching in G

G_M = G with edges of M oriented $R \longrightarrow L$ and all others $L \longrightarrow R$

If we do a BFS of G_M starting from $L \cap F$ we get layers



DEFINITION A blocking set of augmenting paths is a set $\mathcal{P} = \{P_1, \ldots, P_k\}$ s.t.

(i) Each P_i is M-augmenting.

(ii) All edges in all paths in $\mathcal{P}$ are advancing.

(iii) No two paths in $\mathcal{P}$ share a common vertex

(iv) No proper superset of $\mathcal{P}$ has (i)–(iii)

" $\mathcal{P}$ is a maximal set of vertex-disjoint advancing M-augmenting paths."

Algorithm. Initialize $M = \emptyset$

Repeat {

Compute G_M and its BFS-layer decomposition.

DFS-based algorithm achieves this in $O(m)$ time. → Find a blocking set $\mathcal{P}$ of M-augmenting paths $\{P_1, \dots, P_k\}$

$M \leftarrow M \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_k$

} until $\mathcal{P} = \emptyset$.

Bound # of iterations ... how?

Lemma. If M_t and M_{t+1} are the matchings in 2 consecutive iterations of H-K alg., the length of the shortest M_{t+1}-augmenting path is strictly greater than the length of the shortest M_t-augmenting path.

Proof. Let $\mathcal{L}_0, \mathcal{L}_1, \dots$ denote the BFS layers of G_{M_t} and suppose $\mathcal{L}_d$ is lowest numbered layer that intersects $R \cap F$.
(Note d = length of shortest M_t-augmenting path.)

Now suppose $u_0, u_1, \ldots, u_j$ is a shortest M_{t+1}-augmenting path, with $u_0 \in L$ and $u_j \in R$.

① u_0, u_j both free w.r.t. M_t.

② Edges (u_i, u_{i+1}) are all either

a. Advancing edges in G_{M_t} whose orientation is reversed in $G_{M_{t+1}}$ because (u_{i+1}, u_i) belonged to blocking set $P = M_t \oplus M_{t+1}$

$\text{Layer}(u_i) = \text{Layer}(u_{i+1}) + 1$

b. Advancing edges in both G_{M_t} and $G_{M_{t+1}}$,

$\text{Layer}(u_{i+1}) = \text{Layer}(u_i) + 1$

c. Retreating edges in G_{M_t}

$\text{Layer}(u_{i+1}) < \text{Layer}(u_i)$.

$\text{Layer}(u_{i+1}) \leq 1 + \text{Layer}(u_i)$

equality holds only in case b.

$d \leq \text{Layer}(u_j) \leq j$

equality holds only if every edge of $u_0, \ldots, u_j$ satisfies case b.

$d = j$ would contradict maximality of P.

$\Longrightarrow d < j$. ∎

After $t = \frac{1}{2}\sqrt{n}$ iterations of HK Alg, shortest M_t-augmenting path has $> \sqrt{n}$ vertices.

Suppose $M^* =$ max matching and

$|M^*| = |M_t| + k$ ← definition of k.

$M^* \oplus M_t$ contains at least k disjoint M_t-augmenting paths, each having $> \sqrt{n}$ vertices

$\Longrightarrow \quad k\sqrt{n} < n \Longrightarrow k < \sqrt{n}.$

$\Longrightarrow$ at most $\sqrt{n}$ iterations remain.

In total $< \frac{3}{2}\sqrt{n}$ iterations,

$O(m)$ time per iteration

$\Longrightarrow O(m\sqrt{n})$ algorithm.

After t iters, shortest path has $> 2t$ vert.

$2kt \leq n \Longrightarrow k \leq \frac{n}{2t}$

Total iters $\leq t + k \leq t + \frac{n}{2t}$.