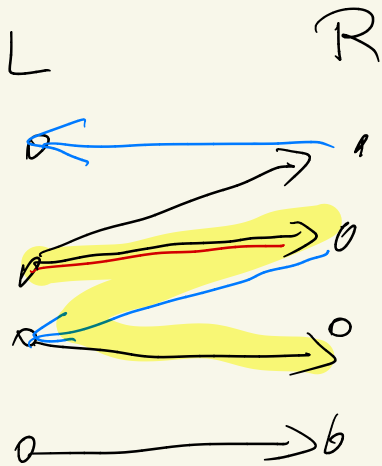
finding matching in bipartite graphs

start $M = \emptyset$

while you can

find augmenting path



matching edge

$R \to L$

all others $L \to R$

augmenting path

= directed path

unmatched node L

to —"— on R

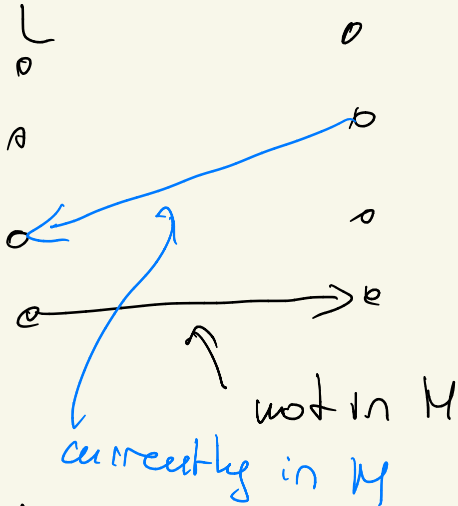
additional input $c_e \geq 0$ all edges

goal: min cost perfect matching

Idea: min augmenting path
look for cheap path

e not in M
cost c_e

e in M
cost $-c_e$



reason: e in path \ M
$\Rightarrow$ we are adding e
e in path $\cap$ M
$\Rightarrow$ we are deleting e

Lemma: M matching P augmenting path

$$M' = M \,\Delta\, P$$

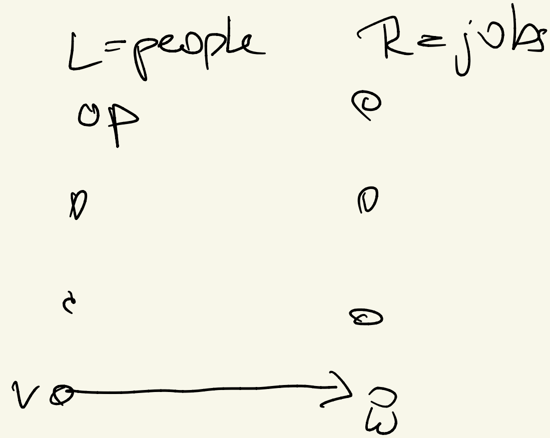
$$\underbrace{\text{cost } M'}_{\sum_{e \in M'} c_e} = \underbrace{\text{cost } M}_{\sum_{e \in M} c_e} + \underbrace{\text{cost of } P}_{\text{signed cost as defined above}}$$

Issues: 1 is outcome optimal

2 can we find shortest path

3 can we use dijkstra?

prices on nodes



new cost $p(v) + c_e - p(w) = c_e^p$

bonus for v

value of w

Claim 1: M is min cost perf M for cost c iff only for cost c^p

$$\sum_{e \in M} c_e^p = \sum_{e=(v,w) \in M} \left(p(v) + c_e - p(w) \right)$$

$$= \sum_{e \in M} c_e + \sum_{e=(v,w) \in M} \left[p(v) - p(w) \right]$$

$$= \sum_{e \in M} c_e + \sum_{v \in L} p(v) - \sum_{w \in R} p(w)$$

Goals:

define p each step so that

(i) $p(v) = 0$ if $v \in L$ & not matched

(ii) $c_e^p \geq 0$

(iii) $c_e^p = 0$ if $e \in M$

if I can do this then:

① perfect matching of p satisfying above $\overset{?}{\Longrightarrow}$ M is min cost

Yes in c^p costs:

our cost of M is 0

all edges ≥ 0

② using Dijkstra (non-negative costs)

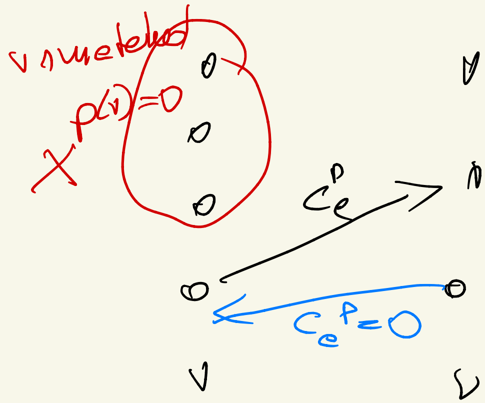
$e \notin M$ $\quad \xrightarrow{c_e^p}$

$e \in M$ $\quad \xleftarrow{-c_e^p}$

all ≥ 0 due to items 2&3

Last part: maintaining p in algorithm

start $p = 0$ all nodes



Dijkstra:

gives cost of min cost path from unmatched nodes to all other nodes

$\text{dist}(X, v)$ = cost unmatched set X on L to node v

update $p'(v) = p(v) + \text{dist}(X, v)$

Claim: satisfies required properties

(i) $p(v) = 0$ if unmatched

$\Rightarrow v \in X$ & $\text{dist}(X, v) = 0$

(ii) $e \notin M \cup M'$ (old or new)

$e = (v, w)$

$v \xrightarrow{e} w$

$\text{dist}(X, w) \leq \text{dist}(X, v) + c_e^p$

see expanded explanation below $\Rightarrow c_e^{p'} \geq 0$

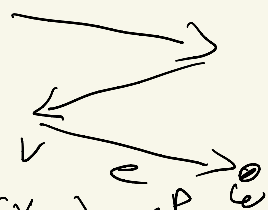
(ii b) $e \in M$ $\quad v \xleftarrow{c_e^p}$

above ineq is now =

$\text{dist}(X, v) = \text{dist}(X, w) - e^p$

(ii c) $e \in M' \setminus M$

select path is min cost



$d(X, w) = \text{dist}(X, v) + c_e^p$

$$d(X,w) \leq \mathrm{dist}(X,v) + c_e^p$$

$$= \mathrm{dist}(X,v) + c_e + p(v) - p(w)$$

$$= p'(v) + c_e - p(w)$$

$$\Longrightarrow \quad p'(v) + c_e - p(w) - d(X,w) \geq 0$$

$$\|$$

$$p'(v) + c_e - p'(w)$$