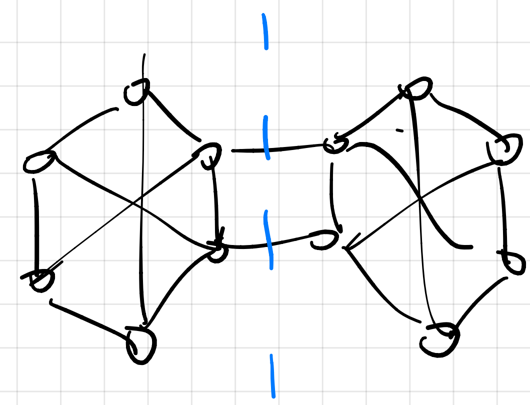
15 Nov 2023

Sparsest Cut

A graph with a sparse cut:



A "cut" in this context is a partition of $V(G)$ into sets A, B, both non-empty.

The sparsity of a cut

$$\frac{\mathrm{cap}(A,B) = \sum_{u \in A, v \in B} c(u,v)}{|A| \cdot |B|}$$

More generally, in the context of a set of vertex pairs we wish to separate, $\{(s_i, t_i) : i = 1, \ldots, k\}$

say $\mathrm{sep}(A,B) := \#\{i \mid s_i \in A, t_i \in B \text{ or } s_i \in B, t_i \in A\}$

and $\mathrm{Sparsity}(A,B) = \dfrac{\mathrm{cap}(A,B)}{\mathrm{sep}(A,B)}$

The definition above corresponds to $k = \binom{n}{2}$ and the set of pairs (s_i, t_i) is all vertex pairs.

This lecture: focus on undirected graphs.

How to verify a graph has no cuts with sparsity $< r$, for some r?

<u>One answer:</u> A <u>concurrent</u> <u>multicommodity flow</u> of value r; is defined to be a k-tuple of flows, $(f_1, \ldots, f_k)$, such that f_i is a flow of value r from s_i to t_i $\forall i$

and, for every edge $e = (u,v)$ of capacity $c(e)$,

$$\sum_{i=1}^{k} |f_i(u,v)| \leq c(e).$$

Suppose (A,B) is a cut and $sep(A,B) = j$,

i.e. for some set $J \subseteq [k]$, with $|J| = j$,

we have $|\{s_i, t_i\} \cap A| = 1$ for all $i \in J$.

Consider the flows $\{ f_i \mid i \in J \}$.

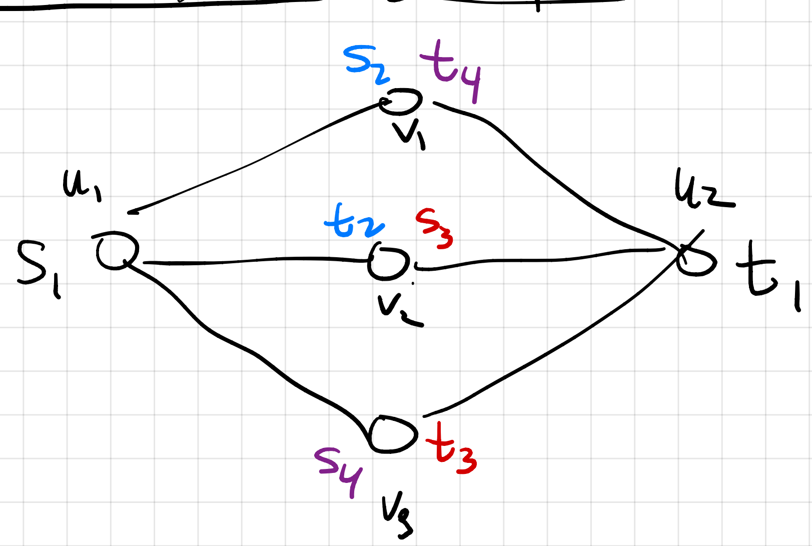
Each has value r, so $|f_i(A,B)| = r$ $\forall i \in J$.

$$\sum_{i \in J} |f_i(A,B)| = j \cdot r$$

$$\text{cap}(A,B) \geq \sum_{i \in J} |f_i(A,B)|, \qquad j \cdot r = sep(A,B) \cdot r$$

$$\frac{\text{cap}(A,B)}{sep(A,B)} \geq r.$$

<u>Okamura – Seymour Example</u>



All edges capacity 1.

$A = \{v_1\}$ $\qquad B = \{u_1, u_2, v_2, v_3\}$

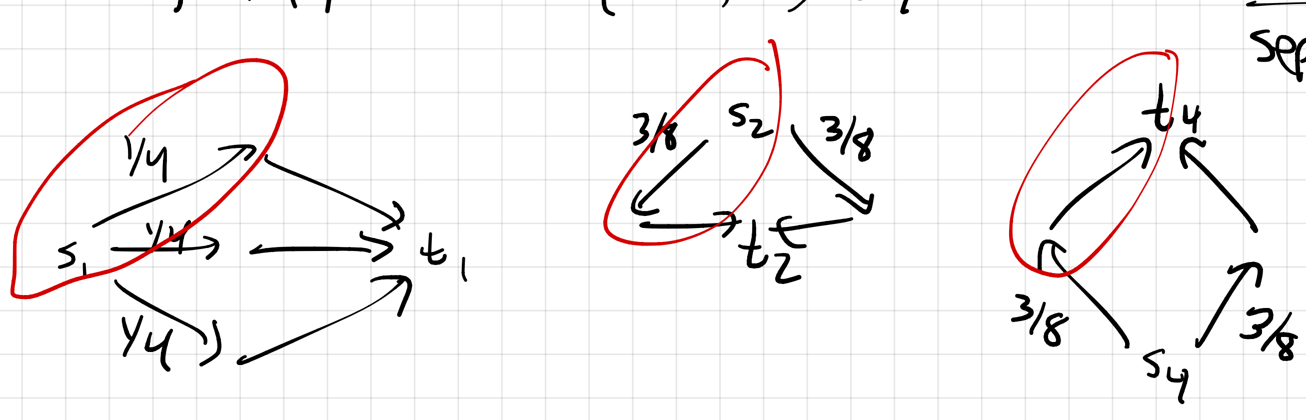
$\text{cap}(A,B) = 2$

$sep(A,B) = 2$

sparsity $= 1$.

$A = \{u_1, v_1\}$ $\qquad B = \{u_2, v_2, v_3\}$

$$\frac{\text{cap}(A,B)}{sep(A,B)} = \frac{3}{3} = 1.$$



Sending from s_i to t_i at ~~rate~~ r

consumes $\geq 2r$ units of capacity.

4 terminal pairs, $2r$ units of cap per terminal pair $\Longrightarrow$ $8r$ units of capacity required.

Graph has only 6 units of capacity total

$\therefore$ $r \leq 3/4$.

The Leighton - Rao Approximate Max-Flow Min-Cut Theorem:

The sparsest cut value of an undirected graph with k terminal pairs exceeds the max concurrent flow rate by $O(\log k)$.

A linear program for concurrent flow:

$$\mathcal{Q} = \{ (P_1, \ldots, P_k) \mid P_i \text{ is a path from } s_i \text{ to } t_i \}$$

For $Q = (P_1, \ldots, P_k) \in \mathcal{Q}$ and $e \in E$

$$n_Q(e) = \# \{ i \mid e \in P_i \}.$$

(PRIMAL)

$$\max \sum_{Q \in \mathcal{Q}} x_Q$$

$$\text{s.t.} \quad \sum_{Q \in \mathcal{Q}} n_Q(e) x_Q \leq c(e) \quad \forall e \quad (y_e)$$

$$x_Q \geq 0 \quad \forall Q$$

(DUAL)

$$\min \sum_{e \in E} c(e) y_e$$

$$\text{s.t.} \quad \sum_{e \in E} n_Q(e) y_e \geq 1 \quad \forall Q$$

$$y_e \geq 0 \quad \forall e$$

If $Q = (P_1, P_2, \ldots, P_k)$

"combined length of $P_1, \ldots, P_k$ when edge lengths are y_e."

$$\sum_{e \in E} n_Q(e) y_e = \sum_{e \in E} \sum_{i \in [k]} \mathbb{1}[e \in P_i] y_e = \sum_{i \in [k]} \sum_{e \in P_i} y_e$$