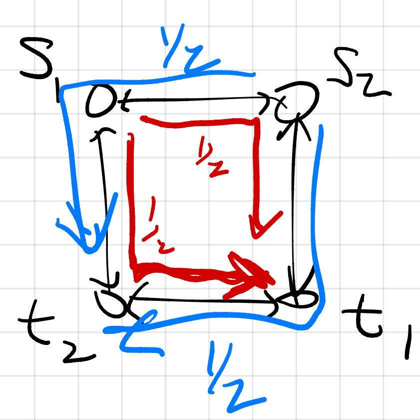
13 Nov 2023 — Multicommodity Flow

For today, all edges have capacity 1.

<u>Problem:</u> Given $G = (V,E)$, along with pairs $(s_1,t_1), \ldots, (s_k,t_k)$.

Pack as many (fractional) paths into G as possible, given each path P must belong to $\mathcal{P} = \{$paths with endpoints s_i, t_i for some $i\}$ and each edge used by ≤ 1 path in total.

<u>Ex.</u>



(PRIMAL)

$$\max \sum_{P \in \mathcal{P}} x_P$$
$$\text{s.t.} \sum_{P:\, e \in P} x_P \leq 1 \quad \forall e$$
$$x_P \geq 0$$

(DUAL)

$$\min \sum_{e \in E} y_e$$
$$\text{s.t.} \sum_{e \in P} y_e \geq 1 \quad \forall P \in \mathcal{P}$$
$$y_e \geq 0$$

Game theoretic interpretation:

simultaneously
- Kicker chooses a src-sink pair s_i, t_i and a path P from s_i to t_i.
- Goalie chooses $e \in E$, wins if $e \in P$.

Algorithm

1. Initialize $w_e^1 = 1 \quad \forall$ edge e // w_e^t is goalie's un-normalized weights at start of round t.

2. for $t = 1, \ldots, T$:

choose P_t to minimize $\sum_{e \in P_t} w_e^t$

for all $e \in E$:

$$w_e^{t+1} = w_e^t \cdot \begin{cases} (1+\varepsilon) & \text{if } e \in P_t \\ 1 & \text{if } e \notin P_t \end{cases}$$

endfor

endfor

output $f_{ALG} = \frac{1}{T} \sum_{t=1}^{T} f^{P_t}$. ← Elementary flow sending one unit on P_t.

... scaled as high as possible without overloading edges.

Analysis. Let $W^t = \sum_{e \in E} w_e^t$ for all t.

$$\frac{1}{T}\sum_{t=1}^{T} \underbrace{\sum_{e \in E} \left(\frac{w_e^t}{W_t}\right) \mathbb{1}[e \in P_t]}_{\text{Goalie exp. payoff @ } t} \geq (1-\varepsilon) \max_{e \in E} \Big\{ \underbrace{\frac{1}{T}\sum_{t=1}^{T} \mathbb{1}[e \in P_t]}_{f_{ALG}(e)} \Big\} - O\left(\frac{\ln m}{\varepsilon T}\right)$$

(by performance guarantee of MW algorithm)

$$\max_{e \in E} \{ f_{ALG}(e) \} \leq \frac{1}{(1-\varepsilon)T} \sum_{t=1}^{T} \boxed{\sum_{e \in E} \left(\frac{w_e^t}{W^t}\right) \mathbb{1}[e \in P_t]} + O\left(\frac{\ln m}{(1-\varepsilon)\varepsilon T}\right)$$

← P_t was chosen to minimize this sum.

If $f_{OPT} = \dfrac{\text{optimum MCF}}{\text{val(optimum MCF)}}$ then $\forall t \; \sum_{e \in E} \left(\frac{w_e^t}{W^t}\right) \mathbb{1}[e \in P_t]$

$\nu^* := $ val(optimum MCF)

$$\leq \sum_{e \in E} \left(\frac{w_e^t}{W^t}\right) f_{OPT}(e)$$

... by greedy property of P_t.

$\forall e \quad f_{OPT}(e) \leq \frac{1}{\nu^*}$

$$\max_{e\in E}\left\{f_{ALG}(e)\right\} \leq \frac{1}{(1-\varepsilon)T}\sum_{t=1}^{T}\sum_{e\in E}\left(\frac{w_e^t}{W_t}\right)f_{OPT}(e) + O\left(\frac{\ln m}{(1-\varepsilon)\varepsilon T}\right)$$

$$\leq \frac{1}{(1-\varepsilon)T}\sum_{t=1}^{T}\sum_{e\in E}\left(\frac{w_e^t}{W^t}\right)\frac{1}{v^*} + O\left(\frac{\ln m}{(1-\varepsilon)\varepsilon T}\right)$$

$$= \frac{1}{(1-\varepsilon)Tv^*}\sum_{t=1}^{T}\Bigg[\underbrace{\sum_{e\in E}\left(\frac{w_e^t}{W^t}\right)}_{=1}\Bigg] + O\left(\frac{\ln m}{(1-\varepsilon)\varepsilon T}\right)$$

$$= \frac{1}{(1-\varepsilon)v^*} + O\left(\frac{\ln m}{(1-\varepsilon)\varepsilon T}\right) \leq \frac{1}{(1-2\varepsilon)v^*}$$

for large enough T.