1 Nov 2023

Finishing MAX CUT
Starting Chernoff Bound

Goemans & Williamson.

Given $G=(V,E)$,

Solve

$$\max \sum_{(i,j)\in E} \frac{1}{2}(1-a_{ij})$$

$$\text{s.t.} \quad A \succeq 0$$

$$a_{ii} = 1 \quad \forall i$$

Find vectors $x_1, \ldots, x_n \in \mathbb{R}^n$ s.t. $a_{ij} = x_i^T x_j$
$\forall (i,j) \in [n]\times[n]$.

Random hyperplane rounding:

pick uniformly random unit vector $w \in \mathbb{R}^n$

$$A = \{ i \mid w^T x_i \geq 0 \}$$

$$B = \{ i \mid w^T x_i < 0 \}$$

Analyze approximation ratio edge-by-edge.

If (i,j) is an edge with vector $x_i, x_j \in \mathbb{R}^n$,

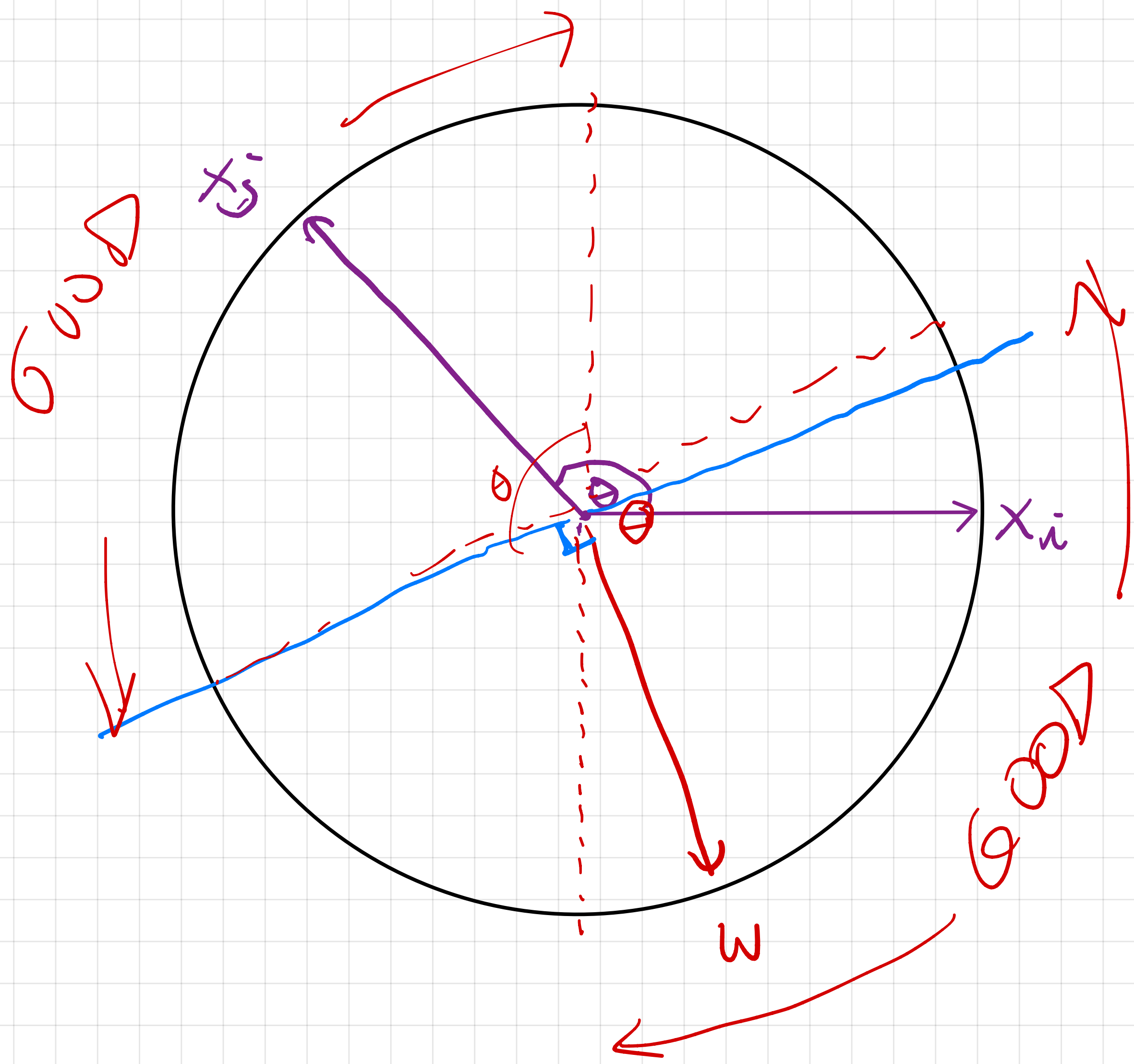
(a) Contribution of (i,j) to SDP opt. value is

$$\frac{1}{2}(1 - x_i^T x_j) = \frac{1}{2}(1 - \cos\theta)$$

(b) Contribution ~~of~~ (i,j) to expected cut value is

$$\Pr(w^T x_i,\ w^T x_j \text{ have opposite signs})$$

By rotation invariance, we just need to answer (b) when $x_i = e_1$, $x_j = (\cos\theta)e_1 + (\sin\theta)e_2$.



$$\Pr(\text{good } w) = \frac{2\theta}{2\pi} = \frac{\theta}{\pi}$$

$$\frac{E(\text{Contrib of } (i,j) \text{ to cut})}{\text{Contrib of } (i,j) \text{ to SDP}} = \frac{\theta/\pi}{\frac{1}{2}(1-\cos\theta)}$$

Approximation ratio of G-W alg $\geq \min_{\theta} \dfrac{\theta/\pi}{\frac{1}{2}(1-\cos\theta)} = 0.878\ldots$

Chernoff Bound

If $X_1, \ldots, X_N$ are independent random variables, each non-negative, and....

(i) each X_i is guaranteed to be small individually

(ii) $\sum X_i$ is expected to be large collectively

then $\Pr(\sum X_i$ is far its expectation$)$ is exponentially small.

Theorem. $X_1, \ldots, X_N$ independent, $[0,1]$-valued.

Let $X = \sum_{i=1}^{N} X_i$ and $\mu = \mathbb{E}[X]$.

For every $\beta > 1$

$$\Pr(X \geq \beta \cdot \mu) < \exp\left(-\mu\left(\beta \ln \beta - \beta + 1\right)\right)$$

For every $0 < \beta < 1$

$$\Pr(X \leq \beta \cdot \mu) < \exp\left(-\mu\left(\beta \ln \beta - \beta + 1\right)\right)$$

If $\beta = 1 + \varepsilon$, $\beta \ln \beta - \beta + 1 > \frac{1}{3}\varepsilon^2$ provided $0 < \varepsilon < 1$.

$\beta = 1 - \varepsilon$, $\beta \ln \beta - \beta + 1 \geq \frac{1}{2}\varepsilon^2$

$$\Pr(X \geq (1+\varepsilon)\mu) < \exp\left(-\frac{\mu}{3}\varepsilon^2\right)$$

$$\Pr(X \leq (1-\varepsilon)\mu) < \exp\left(-\frac{\mu}{2}\varepsilon^2\right)$$

Proof technique. Calculate $\mathbb{E}[e^{tX}]$.

$$\mathbb{E}[e^{tX}] = \mathbb{E}\left[e^{tX_1} \cdot e^{tX_2} \cdots e^{tX_N}\right]$$

$$= \prod_{i=1}^{N} \mathbb{E}[e^{tX_i}].$$

Then use Markov's ineq. on rand var e^{tX} and optimizing over t.