27 Oct 2023

Approx Algs by Rounding Convex Relaxations

Announcement: Class meets in Gates 122 on Monday.

min weight vertex cover: minimize $\sum_{v \in S} w(v)$ over S, a vertex cover of G.

Equivalent integer program

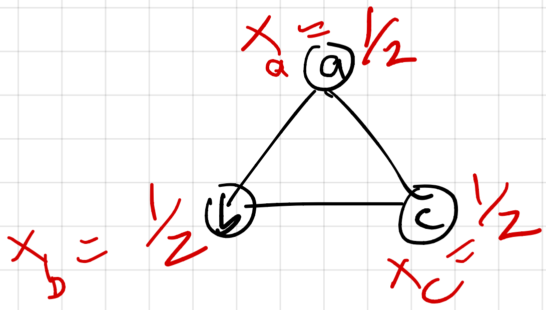
$$\min \sum w(v)\, x_v$$
$$\text{s.t. } x_u + x_v \geq 1 \quad \forall (u,v) \in E$$
$$x_u \in \{0,1\} \quad \forall u \in V$$

LP relaxation

$$\min \sum w(v) x_v$$
$$\text{s.t. } x_u + x_v \geq 1$$
$$0 \leq x_u \cancel{\leq 1} \quad \forall u \in V$$

superfluous

A fractional vertex cover that isn't a true vertex cover:



Any true vertex cover satisfies $\sum_{u \in V} x_u \geq 2$.

But this fractional one has $\sum x_u = 3/2$ so it is not a convex combo of genuine vertex covers.

LP Rounding Algorithm for Vtx Cover

1. Solve LP relaxation
2. Round each x_u to nearest integer. (Round $x_u = \frac{1}{2}$ up to 1.)
3. Output $S = \{ v \mid x_v \text{ rounded to } 1 \}$.

Why is S a vertex cover?

$\forall (u,v) \in E \qquad x_u + x_v \geq 1$

$\max\{x_u, x_v\} \geq 1/2$

$\hookrightarrow$ At least one of x_u, x_v rounds up to 1.

Why is its cost approximately optimal?

For all $x \geq 0$, $\quad \text{ROUND}(x) \leq 2x$.

If S is the set chosen by our algorithm

$$\sum_{v \in S} w(v) = \sum_{v \in V} w(v) \cdot \text{ROUND}(x_v)$$

$$\leq 2 \sum_{v \in V} w(v) x_v = 2 \cdot (\text{LP-OPT})$$

$$\leq 2 \cdot \text{OPT}.$$

OPT is minimizing over integer solutions. LP-OPT minimizes over integer and fractional solutions.

Primal-Dual Approx. Algorithm

Plan of attack: Formulate the dual of the vertex cover relation.

Output: $x \in \{0,1\}^V$ which is feasible for the vertex cover LP

i.e. x is (a vector encoding of) a vertex cover.

$\vec{y}$ which is feasible (not necessarily optimal) for the dual LP.

Algorithm designer ensures this

weak duality

s.t. $\text{LP-OBJ}(x) \leq 2 \cdot \text{DUAL-OBJ}(\vec{y}) \leq 2 \cdot (\text{LP-OPT})$

PRIMAL

$$\min \sum_{v} w(v)\, x_v$$

s.t. $x_u + x_v \geq 1 \quad \forall (u,v) \in E$ $\quad y_{uv}$

$x_v \geq 0 \quad \forall v \in V$

DUAL

$$\max \sum_{(u,v)\in E} y_{uv}$$

s.t. $\sum_{u \in N(v)} y_{uv} \leq w(v) \quad \forall v \in V$

$y_{uv} \geq 0 \quad \forall (u,v) \in E$

$y_{uv} = y_{vu} \quad \forall (u,v) \in E$

$N(v) = \{u \mid u \text{ is adjacent to } v \text{ in } G\}$

Algorithm.

Initialize $\quad x_v = 0 \quad \forall v$

$\qquad y_{uv} = 0 \quad \forall (u,v)$

$\qquad s_v = 0 \quad \forall v \quad$ // $s_v = \sum_{u \in N(v)} y_{uv}$

Mark all edges uncovered.

while $\exists$ edge (u,v) that is uncovered

// Increase y_{uv} as much as possible, respecting dual constraints.

$\delta = \min\{ w(u) - s_u,\ w(v) - s_v \}$

$y_{uv} = \delta$

$s_u \leftarrow s_u + \delta, \qquad s_v \leftarrow s_v + \delta$

for all $g \in \{u,v\}$ s.t. $s_g = w(g)$:

$x_g = 1$

mark all edges incident to g as covered

endwhile

output $S = \{v \mid x_v = 1\}$.

Why is PRIMAL $\leq 2 \cdot$ DUAL?

$$\text{PRIMAL} = \sum_v w(v)\, x_v$$

$$= \sum_v s_v x_v \leq \sum_v s_v$$

This increases by 2δ...

$$\text{DUAL} = \sum_{(u,v)\in E} y_{uv}$$

... whenever this increases by δ.