$\boxed{\text{25 Oct 2023}}$

Approximation Algorithms

For a minimization problem, an α-approximation algorithm computes a number, ALG, such that

$$\forall \text{ input } x \qquad \mathrm{OPT}(x) \le \mathrm{ALG}(x) \le \alpha \cdot \mathrm{OPT}(x)$$

Duh   Aha!

In this type of guarantee, α is often a constant, but could be a function of $n = |x|$ or even of x itself.

Duh   Aha!

For maximization, $\mathrm{ALG}(x) \le \mathrm{OPT}(x) \le \alpha \cdot \mathrm{ALG}(x)$ $(\alpha \ge 1)$

Or sometimes, $\alpha \cdot \mathrm{OPT}(x) \le \mathrm{ALG}(x) \le \mathrm{OPT}(x)$ $(\alpha \le 1)$

Def. A vertex cover of an undirected graph G is (a) a set of vertices, S, such that every edge has an endpoint in S.

(b) the complement of an independent set.

Min cardinality vertex cover

Theorem. (König–Egervary) If G is bipartite,

$$\min\{|S| : S \text{ a vertex cover}\} = \max\{|M| : M \text{ a matching}\}$$

Proof sketch. Apply max-flow min-cut.

Greedy 2-approx alg:

$S = \emptyset$, mark all edges uncovered, $M = \emptyset$

while $\exists$ an uncovered edge $e = (u,v)$:

$S \leftarrow S \cup \{u,v\}$; $M \leftarrow M \cup \{e\}$

mark all edges incident to u & v as covered

endwhile

output S

1. S is a valid vertex cover.
(Every edge e' got covered in the iteration where we marked it as covered.)

2. $|S| \leq 2 \cdot |OPT|$.

By construction $|S| = 2|M|$.

By pigeonhole, $|M| \leq |OPT|$

because if S^* is an opt. vertex cover, there is 1-to-1 mapping $M \to S^*$ that sends $e \in M$, to an endpoint of e.

Randomized 2-approx alg for VC

$S = \emptyset$, mark all edges uncovered

while $\exists$ edge $e = (u,v)$ which is uncovered:

toss a fair coin and:

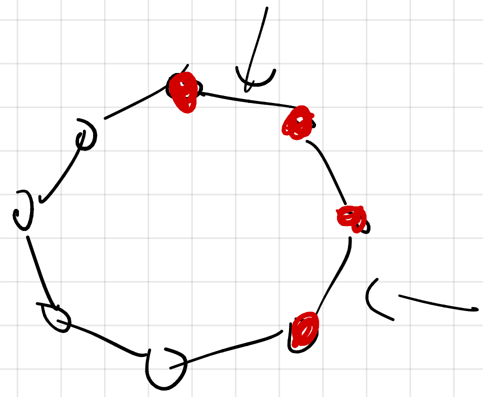
heads $\longrightarrow$ $S = S \cup \{u\}$

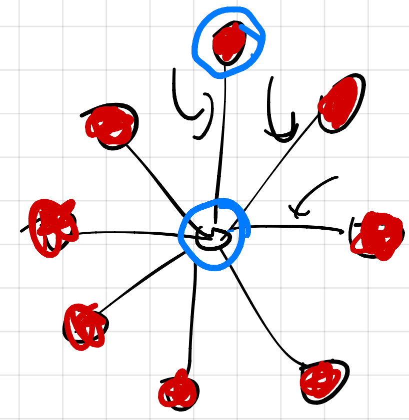
tails $\longrightarrow$ $S = S \cup \{v\}$.

mark all edges incident to the new vertex as covered

endwhile

output S





Claim. $\mathbb{E}|S| \le 2 \cdot OPT$

Proof. Let S^* = any minimum vertex cover

$X_t = |S \cap S^*|$ after t iterations

$Y_t = |S \setminus S^*|$ after t iterations

$$X_t + Y_t = |S| = t \quad \forall t.$$

$$\mathbb{E}[X_{t+1} - X_t] = \begin{cases} 1 & \text{if } \{u,v\} \subseteq S^* \\ 1/2 & \text{if } \{u,v\} \not\subseteq S^* \end{cases}$$

$$\mathbb{E}[Y_{t+1} - Y_t] = \begin{cases} 0 & \text{if } \{u,v\} \subseteq S^* \\ 1/2 & \text{if } \{u,v\} \not\subseteq S^* \end{cases}$$

At termination $\mathbb{E}|S \cap S^*| \ge \mathbb{E}|S \setminus S^*|$

$$\mathbb{E}|S| = \mathbb{E}|S \cap S^*| + \mathbb{E}|S \setminus S^*|$$
$$\le 2 \cdot \mathbb{E}|S \cap S^*| \le 2|S^*|.$$

Weighted Vertex Cover vertices have weights $w(v) \ge 0$.

Minimize $\sum_{v \in S} w(v)$, subject to S being a vertex cover.

1. Reformulate as an integer program.

Introduce "decision variables" x_v such that $x_v = 1$ indicates $v \in S$

$x_v = 0$ indicates $v \notin S$.

$$\begin{aligned} \min \quad & \sum_v w(v)\, x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall\ e=(u,v) \in E \\ & x_v \in \{0,1\} \quad \forall v \end{aligned}$$

2. Relax to a linear program

$$\begin{aligned} \min \quad & \sum_v w(v)\, x_v \\ \text{s.t.} \quad & x_u + x_v \geq 1 \quad \forall\ e=(u,v) \in E \\ & x_v \geq 0 \quad \forall v \end{aligned}$$

3. Solve the linear program.
("Ellipsoid algorithm" does this in poly time.)