23 Oct 2023 — Strong duality

Announcement: Homework 4 due 11/3, 11:59 pm
Think about verifiers!

Take-home midterm will be 11/6–10.

RDK Weds office hrs. shifted 4–5pm
this week + next.

max $2x_1 + 3x_2$

s.t.

$$x_1 + x_2 + w_1 = 8$$
$$2x_1 + x_2 + w_2 = 12$$
$$x_1 + 2x_2 + w_3 = 14$$
$$\vec{x}, \vec{w} \geq 0$$

max $12 + 2x_2 - w_2$

$$w_1 = 8 - x_1 - x_2 = 2 - \tfrac{1}{2}x_2 + \tfrac{1}{2}w_2$$
$$x_1 = 6 - \tfrac{1}{2}x_2 - \tfrac{1}{2}w_2$$
$$w_3 = 14 - x_1 - 2x_2 = 8 - \tfrac{3}{2}x_2 - \tfrac{1}{2}w_2$$
$$\vec{x}, \vec{w} \geq 0$$

Suppose we start from $x_2 = w_2 = 0$ ← {Fixed variables} = $\{x_2, w_2\}$.

Increase x_2 from 0 to 4, which is when $w_1 = 0$, and x_1, w_3 are still ≥ 0.

Finally rewrite everything as linear func of w_1, w_2.
new set of fixed variables.

Iterate pivoting until one of the following things happens.

1. The objective function has a non-positive coeff. on every fixed variable.

   Then the current solution is certifiably optimal.

   Terminate with finite opt value.

2. There's a fixed var. with positive coeff in the objective function.

   For every non-fixed var, its partial derivative w.r.t. this fixed var. is ≥ 0.

   It means we found a ray contained in the feasible set on which the obj. function is unbounded.

   Terminate and report that opt is unbounded.

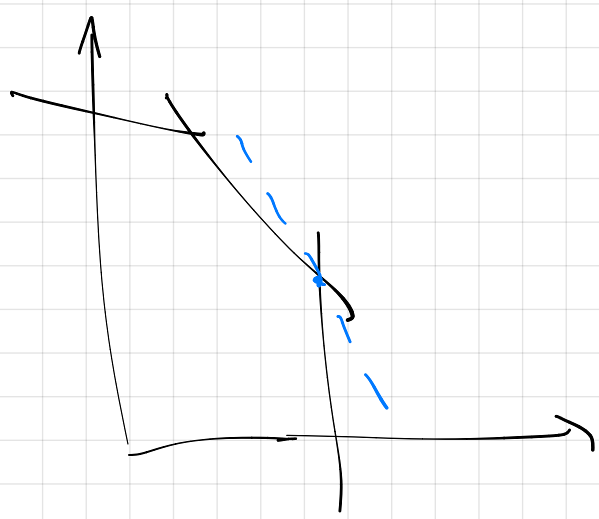
Example:

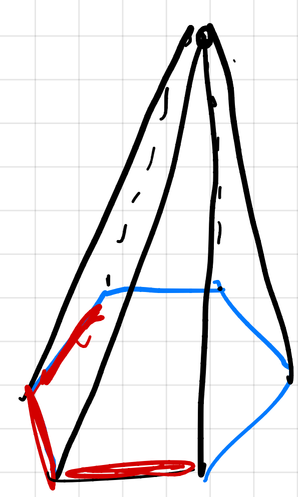
$$\begin{aligned} \max \quad & 18 - x_2 + w_3 \\ \text{s.t.} \quad & x_1 = 5 + x_2 + \tfrac{1}{2} w_3 \\ & w_1 = 3 - 4x_2 + 2w_3 \\ & w_2 = 1 - \tfrac{1}{2}x_2 + \tfrac{1}{3} w_3 \end{aligned}$$

Degenerate pivots.

Example:

$$\begin{aligned} \max \quad & 18 - x_2 + w_3 \\ \text{s.t.} \quad & x_1 = + x_2 - w_3 \\ & w_1 = 3 - 4x_2 + 2w_3 \\ & w_2 = 1 - \tfrac{1}{2}x_2 + \tfrac{1}{3} w_3 \end{aligned}$$





Termination at a finite objective value means: objective function is now written as

$$\text{obj} = v - z^T x - y^T w \qquad z, y \leq 0$$

and we found a feasible point where

$$z^T x = y^T w = 0.$$

The equation $c^T x = \text{obj} = v - z^T x - y^T w$ means that

$$c^T x = v - z^T x - y^T w$$

holds for all x, w satisfying $Ax + w = b$.

In other words

$$c^T x = v - z^T x - y^T(b - Ax)$$

is valid $\forall x \in \mathbb{R}^n$.

$\therefore$

$$0 = v - y^T b$$

$$c^T = -z^T + y^T A$$

We've got $y, z \succeq 0$ s.t.

$$v = b^T y$$

$$A^T y = c + z \implies A^T y \succeq c$$

Dual was

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \succeq c \\ & y \succeq 0 \end{aligned}$$