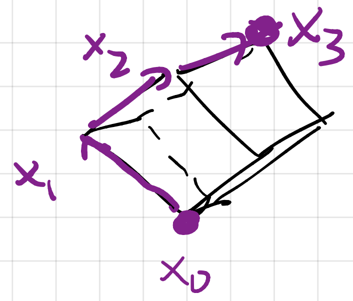
$\boxed{\text{20 Oct 2023}}$ Simplex Algorithm and Strong Duality

$$\begin{array}{llll} \max & c^T x & \min & b^T y \\ \text{s.t.} & Ax \preceq b & \text{s.t.} & A^T y \succeq c \\ & x \succeq 0 & & y \succeq 0 \end{array}$$

The solution set of $\{Ax \preceq b, \; x \succeq 0\}$ is a polyhedron contained in the positive orthant, $\mathbb{R}^n_{\geq 0}$.

Vertices of the polyhedron correspond to n-tuples of (linearly independent) tight constraints.

Simplex method is a local search algorithm that starts at any vertex and walks to a neighboring vertex where the objective function is (weakly) greater, until it cannot find any such neighbor.

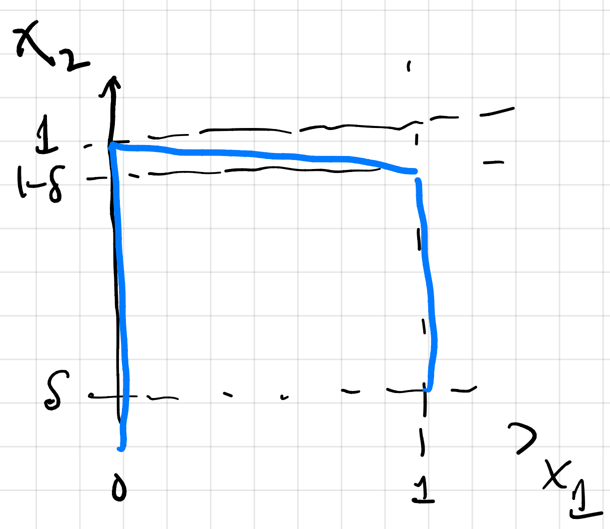


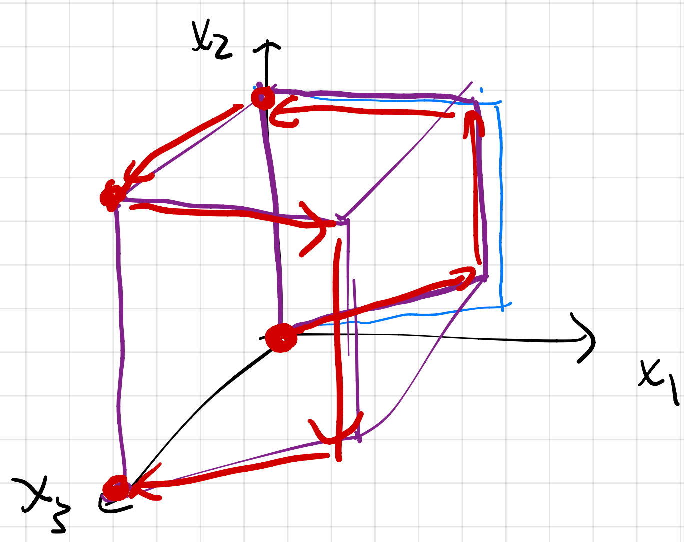
A polyhedron in n dimensions defined by m constraints (plus non-negativity inequalities) has $\leq \binom{m+n}{n}$ vertices.

Klee-Minty cubes: $\max \; x_n$

s.t. $0 \leq x_1 \leq 1$

$\delta x_i \leq x_{i+1} \leq 1 - \delta x_i \quad \forall i < n.$





Equational form:

$$\max \; c^T x \quad \text{s.t.} \; Ax \preceq b, \; x \succeq 0 \qquad \equiv \qquad \max \; c^T x \quad \text{s.t.} \; Ax + w = b, \; x, w \succeq 0$$

vector of "slack variables"

$$\max \; 2x_1 + 3x_2$$

s.t.

$$x_1 + x_2 \le 8$$
$$2x_1 + x_2 \le 12$$
$$x_1 + 2x_2 \le 14$$
$$x_1, x_2 \ge 0$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \preceq \begin{bmatrix} 8 \\ 12 \\ 14 \end{bmatrix}$$

$$\max \; 2x_1 + 3x_2$$

s.t.

$$x_1 + x_2 + w_1 = 8$$
$$2x_1 + x_2 + w_2 = 12$$
$$x_1 + 2x_2 + w_3 = 14$$
$$\vec{x}, \vec{w} \succeq 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{x} \\ \vec{w} \end{bmatrix} = \vec{b}$$

Suppose we start from $x_2 = w_2 = 0$.