$\boxed{\text{18 Oct 2023}}$ Linear Programming

$$\begin{array}{cc} & n \text{ columns} \\ m \text{ rows} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix} \end{array}$$

$Ax \leqslant b$ means A is a matrix

x and b are vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$

and each coord of Ax is $\leq$ the corresponding coordinate of b.

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

$A \in \mathbb{R}^{m\times n}$, often will assume $A \in \mathbb{Z}^{m\times n}$.

$x \in V^n$, $b \in V^m$ where V is an ordered vector space over $\mathbb{R}$.

Primarily $V = \mathbb{R}$.

Every element of V is exactly one of:

- zero (only $\vec{0}$)
- positive
- negative

These satisfy:

- $x \neq 0$ is positive if and only if $-x$ is negative
- positive + positive = positive
- x positive, $a \in \mathbb{R}$ positive, $\Rightarrow ax > 0$.

LP is any of the following:

1. LP feasibility: given A, b, is the solution set of $Ax \leqslant b$ non-empty?
2. LP optimization: given $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ what is the maximum of $c \cdot x$ over all

x satisfying $Ax \preceq b$?

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \end{aligned}$$

... or conclude $Ax \preceq b$ infeasible

... or conclude $c \cdot x$ is unbounded on $\{x \mid Ax \preceq b\}$.

③ LP search: given an LP feasibility or LP optimization problem, find the x that solves it.

LP's and their duals.

1. A LP is in standard form if it is one of the following two types:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \succeq b \\ & x \succeq 0 \end{aligned}$$

$A(x - x') \preceq b$

$|||$

$$\begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} x \\ x' \end{bmatrix} \preceq b$$

For any LP in standard form, it has a dual also in standard form.

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \preceq b \\ & x \succeq 0 \end{aligned} \quad \xleftrightarrow{\text{dual}} \quad \begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \succeq c \\ & y \succeq 0 \end{aligned}$$

Weak Duality. If y satisfies the constraints $\{A^T y \succeq c, y \succeq 0\}$

then $b^T y \geq \max\{c^T x \mid Ax \preceq b, x \succeq 0\}$

Proof.

$$y_1 (Ax)_1 = y_1 (a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n) \leq b_1 \cdot y_1$$

$$\vdots$$

$$y_m (Ax)_m = y_m (a_{m1} x_1 + \dots + a_{mn} x_n) \leq b_m \cdot y_m$$

$$y^T A x \leq b^T y$$

$$(y^T A)\, x \leq b^T y$$

$$(A^T y)^T x \leq b^T y$$

$$\underbrace{(A^T y - c)^T}_{\geq 0} \underbrace{x}_{\geq 0} + c^T x \leq b^T y$$

$$c^T x \leq b^T y$$

Strong Duality: For any pair of primal & dual LP's in standard form, exactly one of 3 cases applies:

1. $\max\ c^T x$ s.t. $Ax \leq b$, $x \geq 0$, $\min\ b^T y$ s.t. $A^T y \geq c$, $y \geq 0$ are finite and equal.

2. primal infeasible, dual unbounded

3. primal unbounded, dual infeasible