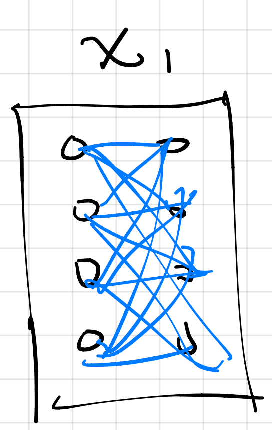
16 Oct 2023 Max-Cut

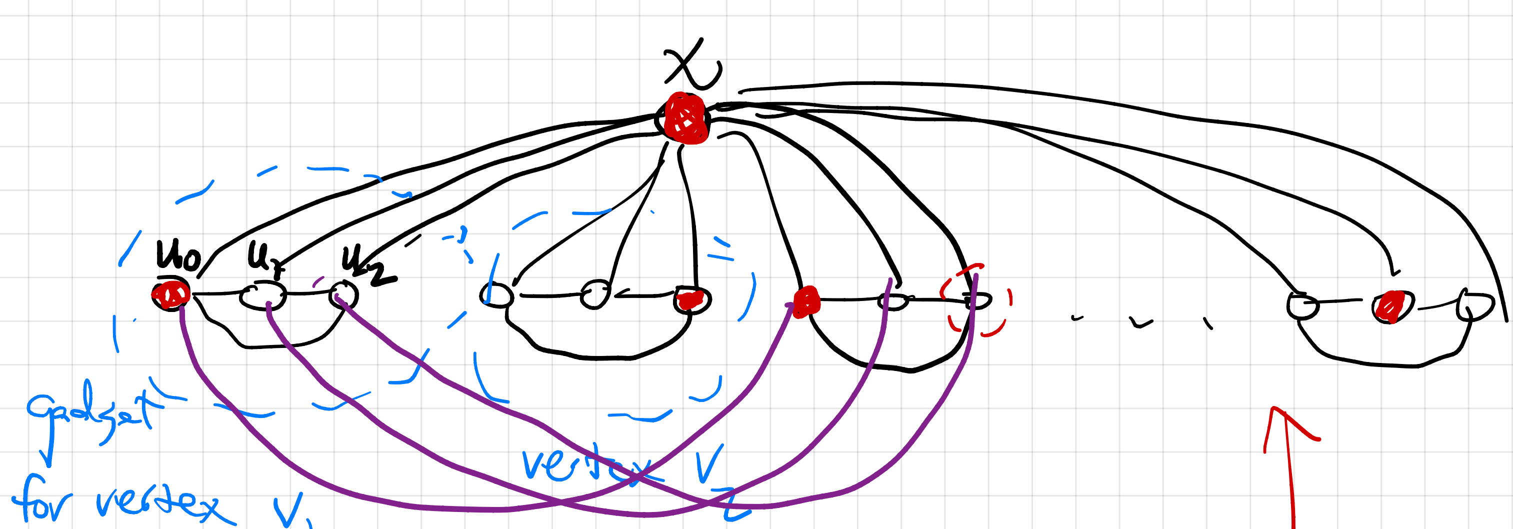
The MAX-CUT problem is: given graph G (undirected) and integer $k > 0$, can the vertex ~~set~~ $V(G)$ be partitioned into A, B such that at least k edges are "cut" by the partition. (Meaning one endpoint in A, the other in B.)

How to prove MAX-CUT is NP-hard?

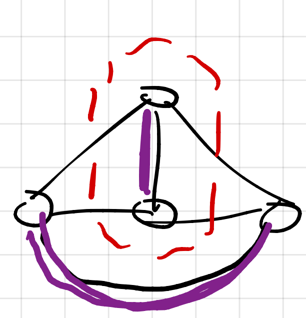


A promising gadget for a 3SAT reduction, but it becomes tricky to represent clauses.

For reducing 3-COLORABILITY $\leq_p$ MAX-CUT....



Cuts in this graph that cut $4n$ edges are in bijection with 3-colorings of the vertices.



Proposed Reduction: Given graph H which is an instance of 3-COLORABILITY, construct G whose vertex set is:

- $\{x\}$ (root vertex)
- $\{u_0, u_1, u_2\}$ (3 gadget nodes corresponding to each $u \in V(H)$)

and edge set is

- Edges from x to every other vertex
- Edges (u_i, u_j) for all $u \in V(H)$ and $i \neq j$ in $\{0,1,2\}$.
- Edges (u_i, v_i) for all $(u,v) \in E(H)$ and $i \in \{0,1,2\}$.

Finally, we ask if $\exists$ a cut of size $4n + 2m$ edges. ($n = \#$ vertices in H, $m = \#$ edges in H)

Conclusion:

- Cuts that cut $> 4n$ black edges don't exist
- Cuts that cut exactly $4n$ black edges and at least $2m$ purple edges do correspond to proper 3-colorings of H.
- Cuts that cut $< 4n$ black, but $\geq 4n + 2m$ edges in total may exist,

violating correctness of reduction.

If black edges have weight $w > 3m$ (total # purple edges) then cuts of total weight $4nw + 2m$ <u>must</u> cut $4n$ black edges and, in addition, at least $2m$ purples.

<u>Conclusion 2.</u> There is a slightly more general problem, WEIGHTED MAX CUT, where the input is:

- graph G
- edge weights $w(u,v) \in \mathbb{N}$
- target cut weight K

and the question is: does $\exists$ a vertex partition A, B s.t.

$$\sum_{u \in A} \sum_{v \in B} w(u,v) \geq K \, ?$$

Then the reduction above, with black edge weights set to $w = 3m+1$ and purple edge weights set to 1 and $K = 4nw + 2m$, is a valid reduction

3-COLORABILITY $\leq$ WEIGHTED MAX-CUT

If we allow multigraphs (i.e., where two vertices can potentially be connected by multiple edges) then we can represent an edge of weight w by w parallel edges, and the reduction above becomes a reduction

$$\text{3-COLORABILITY} \le_p \text{MULTIGRAPH MAX-CUT}.$$

If we want our reduction to output a simple graph, we require one more gadget. As before, let

H = instance of 3-COLORABILITY
n = # vertices of H
m = # edges of H
$w = 3m+1$.

and now set

$$M = 6nw + 3m + 1$$

Replace each vertex (other than x) in the original reduction with $2M$ vertices forming a complete bipartite graph with M vertices on each side.

Thus, our reduction takes H and outputs a graph G with

- root vertex x
- vertices $u_{i,j}$ for all $u \in V(H)$, $i \in \{0,1,2\}$, $j \in \{1,2,3,\ldots,2M\}$
- blue edges $(u_{i,j}, u_{i,k})$ whenever $u \in V(H)$, $i \in \{0,1,2\}$, $j-k$ odd
- **black** edges $(x, u_{i,2j})$ for $i \in \{0,1,2\}$ and $1 \le j \le w$.
- **black** edges $(u_{i,2j}, u_{\ell,2j})$ for distinct $i, \ell \in \{0,1,2\}$ and $1 \le j \le w$
- purple edges $(u_{i,1}, v_{i,1})$ whenever $(u,v) \in E(H)$ and $i \in \{0,1,2\}$.

This graph G has $3nM^2$ blue edges, $6nw$ black edges, $3m$ purple edges.

MAX-CUT: Is there a partition that cuts at least $K = 3nM^2 + 4nw + 2m$ edges of G?

Def. A partition of the set

$$V_{u,i} = \{ u_{i,j} \mid 1 \le j \le 2m \}$$

into sets A_{ui}, B_{ui} is "pure" if either

$$A_{ui} = \{ u_{i,j} \mid j \text{ odd} \}, \quad B_{ui} = \{ u_{i,j} \mid j \text{ even} \}$$

or

$$A_{ui} = \{ u_{i,j} \mid j \text{ even} \} \quad B_{ui} = \{ u_{i,j} \mid j \text{ odd} \}$$

A partition ~~of~~ of $V(G) = \{x\} \cup \bigcup_{\substack{u \in V(H) \\ i \in \{0,1,2\}}} V_{u,i}$

into sets A, B is pure if the partition $A \cap V_{u,i}$, $B \cap V_{u,i}$ is pure for all u, i.

Lemma. A pure partition cuts $3nM^2$ blue edges. Any other partition cuts at most $3nM^2 - M$ blue edges.

Proof. Every blue edge has endpoints $u_{i,j}$ and $u_{i,k}$ for some $u \in V(H)$,

$i \in \{0,1,2\}$, and $j,k \in [2M]$ with $j-k$ odd. By definition a pure partition cuts every such edge.

If a partition A, B is not pure, then $\exists u, i$ such that (A_{ui}, B_{ui}) is not a pure partition of V_{ui}. If one of A_{ui}, B_{ui} is empty, then none of the blue edges in $G[V_{ui}]$ is cut, so at least M^2 blue edges are uncut. If A_{ui}, B_{ui} are both non-empty but (A_{ui}, B_{ui}) is not pure, then at least one of the sets $\{u_{ij} \mid j \text{ odd}\}$ or $\{u_{ij} \mid j \text{ even}\}$ intersects both A_{ui} and B_{ui}. Assume WLOG that it is $\{u_{ij} \mid j \text{ odd}\}$. There are M vertices in $\{u_{ij} \mid j \text{ even}\}$ and each of them has ≥ 1 blue edge to another vertex on its side of the partition. Hence at least M blue edges are uncut. $\boxtimes$

Back to analyzing the reduction.

If H is 3-colorable and

$c: V(H) \to \{0,1,2\}$ is a valid coloring

let

$$A = \{x\} \cup \{u_{i,j} \mid u \in V(H),\ i = c(u),\ j \text{ even}\}$$
$$\cup \{u_{i,j} \mid u \in V(H),\ i \neq c(u),\ j \text{ odd}\}$$

$$B = V(G) \setminus A$$

Then the partition (A, B) cuts:

- all blue edges
- $4w$ black edges in each induced subgraph
  $G[\{x\} \cup V_{u0} \cup V_{u1} \cup V_{u2}]$, $u \in V(H)$
- $2m$ purple edges.

hence K edges in total.

Conversely, if $\exists$ a partition (A, B) that cuts K edges in total:

1. The partition must be pure. Otherwise there are at least M uncut blue edges. By the choice of $M > 6nw + 3m$ the combined number of cut black and purple edges can't possibly compensate the loss of M blue edges from the cut.

2. For every $u \in V(H)$, exactly one $i \in \{0, 1, 2\}$ satisfies $\{u_{ij} \mid j \text{ even}\} \subset A$. Otherwise at most $3w$ of the black edges in induced subgraph
$$G\left[\{x\} \cup V_{u0} \cup V_{u1} \cup V_{u2}\right]$$
are cut. By choice of $w > 3m$, no number of cut purple edges can compensate the loss of w

black edges from the cut.

3. Color each vertex $u \in V(H)$ with the unique color $c(u) = i$ such ~~that~~ $\{u_{i,j} \mid j \text{ even}\} \subset A$.

This must constitute a proper coloring of H, as otherwise fewer than $2m$ purple edges would be cut.