4 Oct 2023

PUSH–RELABEL ALGORITHM

initialize $h(s) = n$, $h(v) = 0$ $\forall v \neq S$.

initialize $f(u,v) = \begin{cases} c(u,v) & \text{if } u = s \\ -c(u,v) & \text{if } v = s \\ 0 & \text{otherwise.} \end{cases}$

while $\exists$ $v \neq s,t$ with $x(v) > 0$:

let $u \neq t$ be a vertex of positive excess with maximum height among all such vertices.

if $\exists$ edge (u,v) with $c(u,v) > f(u,v)$ and $h(u) > h(v)$:

// PUSH(u,v)

let $\delta = \min\{x(u), c(u,v) - f(u,v)\}$

$f(u,v) \leftarrow f(u,v) + \delta$

$f(v,u) \leftarrow f(v,u) - \delta$

$\delta = c(u,v) - f(u,v)$ "saturating push"

$\delta < c(u,v) - f(u,v)$ "non-saturating push

else:

// RELABEL(u)

$h(u) \leftarrow h(u) + 1$

endwhile

output f.

Lemma. If f is a preflow and $v_0 \neq s$ is a vertex with $x(v_0) > 0$ then G_f contains a path from v_0 to s made up of edges with $c(u,v) - f(u,v) > 0$.

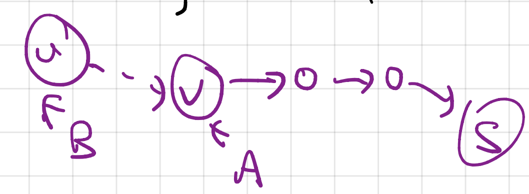
Proof. Let $E_f^+ = \{(u,u') \mid c(u,u') - f(u,u') > 0\}$

and $A = \{w \mid \text{there is a path from } w \text{ to } s \text{ with edges in } E_f^+\}$

$B = V \setminus A$.

Lemma asserts $v_0 \in A$

By construction $\nexists\ (u,v) \in E_f^+$ with $u \in B,\ v \in A$.



$$\sum_{v \in B} x(v) = \sum_{v \in B} \sum_{u \in V} f(u,v)$$

$$= \sum_{u \in V} \sum_{v \in B} f(u,v)$$

$$= \sum_{u \in A} \sum_{v \in B} f(u,v) + \sum_{u \in B} \sum_{v \in B} f(u,v)$$

0 by skew symmetry

$$= -\sum_{v \in B} \sum_{u \in A} f(v,u)$$ ← Edges from B to A are saturated.

$$= -\sum_{v \in B} \sum_{u \in A} c(v,u) \leq 0$$

$s \in A$, so $x(v) \geq 0$ for all $v \in B$.

But $\sum_{v \in B} x(v) \leq 0$ so must be that $x(v) = 0\ \forall v \in B$.

$\therefore\ v_0 \in A$ because $x(v_0) > 0$.

QED.

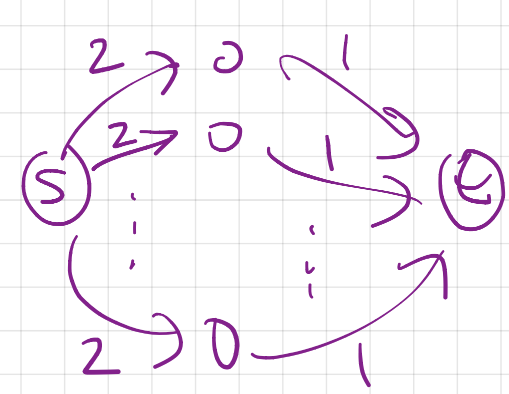
<u>Cor.</u> At all times $\forall v$, $h(v) < 2n$.

<u>Proof.</u> The most recent time we relabeled v, $x(v)$ was > 0. By Lemma, G_f had a path from v to s with pos. resid. cap. on every edge.

$$(v) \to (w) \to \circ \to \circ \to \circ \to (s)$$

$\leq 2n-2$ $\quad \leq 2n-2 \ \ldots \quad \leq n+2 \quad \leq n+1 \quad h(s) = n$

or else we wouldn't have relabeled.



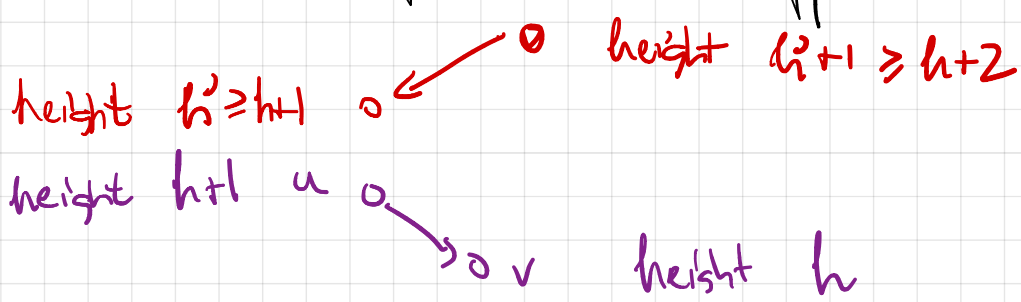
\# relabels $\leq (2n-1)\,n$.

Saturating pushes.

When $\mathrm{PUSH}(u,v)$ saturates (u,v) it means residual cap. of (u,v) becomes equal to zero.

Also, $h(u) = h(v) + 1$

We will not $\mathrm{PUSH}(u,v)$ again until (u,v) has positive residual cap, which happens after $\mathrm{PUSH}(v,u)$.



Between any two saturating pushes of (u,v) $h(v)$ increases by $2 \Longrightarrow$ at most n saturating pushes per (oriented) edge.

$\leq 2mn$ saturating pushes total.

Non-saturating pushes

Let $H = \max_{v \neq s:\, x(v) > 0} \{h(v)\}$.

Divide execution into phases when H is constant

H can only increase during relabels. ($\leq (2n-1)n$ times)

$\#$ phases $\leq 2(2n-1)n$.

Vertices $v \neq s$ will be the source of a non-saturating push at most once per phase.

$\Longrightarrow \leq 2(2n-1)n(n-1)$ non-saturating pushes.