2 Oct 2023 — Push-Relabel Algorithm

Announcement:

- Midterm will be take-home test, 6–10 Nov. (You choose a 48-hr subinterval.)

Def. A preflow with vertex set V satisfies

(i) [skew-symmetry] $f(u,v) + f(v,u) = 0 \quad \forall u,v$

(ii) [semi-conservation] $\underbrace{\sum_{u \in V} f(u,v)}_{x(v) =} \geq 0 \quad \forall v \neq s$

called the "excess of v."

Aside: a circulation is $f: V^2 \to \mathbb{R}$ satisfying $\sum_{u \in V} f(u,v) = 0 \quad \forall v \in V$, including $v = s, t$.

A preflow is feasible if it also satisfies

(iii) [capacity] $f(u,v) \leq c(u,v) \quad \forall u,v.$

The push-relabel algorithm also uses a function $h: V \to \mathbb{N}$ ("height function") satisfying:

(iv) [boundary condition] $h(s) = n, \quad h(t) = 0$

(v) [steepness condition] If $c(u,v) - f(u,v) > 0$ then $h(u) \leq h(v) + 1$.

If P is a simple path from s to t then P contains $\leq n$ vertices, $\leq n-1$ edges.

In $\leq n-1$ hops, P goes from height n to height $\emptyset$, $\therefore$ at least one edge (u,v) in P satisfies $h(u) - h(v) > 1$.

By steepness condition, this (u,v) must be saturated.

If $S = \left\{ \begin{array}{l} \text{vertices reachable from } s \text{ using a path} \\ \text{of edges with } c(u,v) - f(u,v) > 0 \end{array} \right\}$

$T = V \setminus S$

then (S,T) is a cut made up of saturated edges.

Corollary: If f is a flow and h is a height function satisfying boundary and steepness conditions w.r.t. f, then f is a maximum flow.

PUSH-RELABEL ALGORITHM

initialize $h(s) = n$, $h(v) = 0 \quad \forall v \neq s$.

initialize $f(u,v) = \begin{cases} c(u,v) & \text{if } u = s \\ -c(u,v) & \text{if } v = s \\ 0 & \text{otherwise.} \end{cases}$

while $\exists\ v \neq s,t$ with $x(v) > 0$:

let $u \neq t$ be a vertex of positive excess with maximum height among all such vertices.

if $\exists$ edge (u,v) with $c(u,v) > f(u,v)$ and $h(u) > h(v)$:

// PUSH(u,v)

let $\delta = \min\{x(u), c(u,v) - f(u,v)\}$

$f(u,v) \leftarrow f(u,v) + \delta$

$f(v,u) \leftarrow f(v,u) - \delta$

$\delta = c(u,v) - f(u,v)$ "saturating push"

$\delta < c(u,v) - f(u,v)$ "non-saturating push"

else:

// RELABEL(u)

$h(u) \leftarrow h(u) + 1$

endwhile

output f.

Correctness. Inductively verify that properties (i)–(v) above are loop invariants. Then apply Corollary.

Running time: can be implemented to do $O(1)$ operations per while-loop iteration.

Lemma. If f is a preflow, $v \neq s$ is a vertex with $x(v) > 0$, then G_f contains a path from v to s made up of edges with positive residual capacity.

Proof. Let $A = \{ u \mid G_f \text{ contains a } u \to s \text{ path with positive residual cap. edges} \}$.

$$B = V \setminus A$$