

27 Sep 2023

Max-Flow Min-Cut Theorem Ford-Fulkerson Algorithm

Announcements.

- ① Homework 3 to be released Fri, due a week from Fri, shorter than usual.
- ② Email me and Shawn (rdk2, 50396) if you must switch groups.
- ③ In class midterm: I'm considering replacing it with a take-home, the week of Nov 6-10. (Flexible start/end.) Please email me if you're not happy about this.

Recap.

Flow satisfies

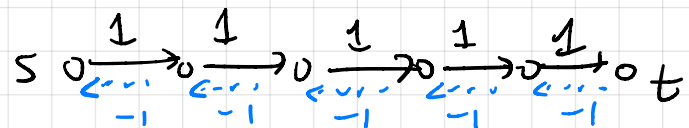
$$f(u,v) + f(v,u) = 0$$

$$\sum_{v \in V} f(u,v) = 0 \quad \forall u \notin \{s,t\}$$

"Feasible"

$$f(u,v) \leq c(u,v) \quad \forall (u,v)$$

Elementary flow f^P



Residual graph G_f

f feasible flow in G .

Capacities

$$c_f(u,v) = c(u,v) - f(u,v)$$

Augmenting path:

path from s to t whose edges have strictly positive residual capacity.

Def. An s - t cut in flow network $G = (V, s, t, c)$ is a partition of V into S, T with $s \in S, t \in T$.

For vertex sets Q, R let

$$f(Q,R) = \sum_{u \in Q} \sum_{v \in R} f(u,v) \quad \text{net flow } Q \rightarrow R$$

$$c(Q,R) = \sum_{u \in Q} \sum_{v \in R} c(u,v) \quad \text{aggregate capacity } Q \rightarrow R$$

Observe: (a) R_1, R_2 disjoint $\Rightarrow f(Q, R_1 \cup R_2) = f(Q, R_1) + f(Q, R_2)$

(b) $f(Q, Q) = 0 \quad \forall Q \subseteq V$

... by skew-symmetry.

Lemma. If f is any flow and S, T is any s-t cut,

$$f(S, T) = \text{val}(f).$$

If f feasible,

$$\text{val}(f) \leq c(S, T)$$

and equality holds if and only if $f(u, v) = c(u, v)$ for all $u \in S, v \in T$. ("S, T is saturated by f.")

Proof. By properties (a), (b) above,

$$f(S, T) = f(S, T) + f(S, S) = f(S, T \cup S) = f(S, V)$$

$$= \sum_{u \in S} \left(\sum_{v \in V} f(u, v) \right) \text{ inner sum equals zero except when } u = s.$$

$$= \sum_{v \in V} f(s, v) = \text{val}(f).$$

Inequality $\text{val}(f) \leq c(S, T)$ follows from $\text{for } u \in S, v \in T$

$$\forall u, v \quad f(u, v) \leq c(u, v)$$

at least one of these is strict <

this is strict <

$$\text{val}(f) = f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) \leq \sum_{u \in S} \sum_{v \in T} c(u, v) = c(S, T)$$

Theorem. (Max-flow Min-cut) For a flow network G

and a feasible flow f , TFAE:

- (i) f is a maximum flow
- (ii) there is no augmenting path in G_f
- (iii) there exists a s-t cut with $c(S, T) = \text{val}(f)$
- (iv) there exists a minimum s-t cut with $c(S, T) = \text{val}(f)$.

Proof. First (iv) \Rightarrow (iii) obvious. To prove (iii) \Rightarrow (iv)

assume f, S, T satisfy (iii) and assume

S^*, T^* is any st cut of minimum capacity.

$$c(S^*, T^*) \leq c(S, T) \quad (\text{def of } S^*, T^*)$$

$$c(S, T) = \text{val}(F) \leq c(S^*, T^*)$$

lemma above

Hence $c(S^*, T^*) = c(S, T)$ are equal, so S, T is a minimum st cut satisfying $\text{val}(F) = c(S, T)$ as required by (iv).

For (i) \Rightarrow (ii) we prove $(\neg \text{ii}) \Rightarrow (\neg \text{i})$.

If G_f has aug path P , let

$$\delta(P) = \min \{ c(u,v) - f(u,v) \mid (u,v) \text{ an edge of } P \}$$

Then $f + \delta(P) \cdot f^P$ is ab. a feasible flow, its value is $\text{val}(F) + \delta(P) > \text{val}(F)$, so f is not a max flow.

For (ii) \Rightarrow (iii): define an augmenting walk to be a

sequence $s = u_0, u_1, u_2, \dots, u_k$ of vertices, st. residual cap > 0 for all (u_i, u_{i+1}) , $0 \leq i < k$.

Let $S = \{ u \mid \exists \text{ an augmenting walk ending at } u \}$

$$T = V \setminus S$$

Note $s \in S$ because (s) is augmenting walk

$t \in T$ because \nexists augmenting path in G_f .

Every (u,v) with $u \in S, v \in T$ has zero residual capacity. This is because

\exists augmenting walk $s = u_0, u_1, \dots, u_k = u$

but $u_0, u_1, \dots, u_k, u_{k+1} = v$ is not an augmenting walk. $\Rightarrow (u,v)$ has ≤ 0 residual capacity. $\therefore = 0$

We have an st cut which is saturated by f , so $c(S, T) = \text{val}(F)$ by lemma above.

Lastly, for (iii) \Rightarrow (i):

If F is feasible flow,

S, T is s-t cut

and $\text{val}(F) = c(S, T)$

} condition (iii)

then for any feasible flow f^* ,

$$\text{val}(f^*) \leq c(S, T) = \text{val}(F)$$

↑
by Lemma

\therefore $\text{val}(F)$ is the maximum value of
a feasible flow in G .