27 Sep 2023

Max-Flow Min-Cut Theorem
Ford-Fulkerson Algorithm

Announcements.

1. Homework 3 to be released Fri, due a week from Fri, shorter than usual.

2. Email me and Shawn (rdk2, so396) if you must switch groups.

3. In class midterm: I'm considering replacing it with a take-home, the week of Nov 6-10. (Flexible start/end.) Please email me if you're not happy about this.

Recap. Flow satisfies $f(u,v) + f(v,u) = 0$

$$\sum_{v \in V} f(u,v) = 0 \quad \forall u \notin \{s,t\}.$$

"Feasible" $\quad f(u,v) \leq c(u,v) \quad \forall (u,v)$

Elementary flow f^P $\quad s \; \circ \underset{-1}{\overset{1}{\rightleftarrows}} \circ \underset{-1}{\overset{1}{\rightleftarrows}} \circ \underset{-1}{\overset{1}{\rightleftarrows}} \circ \underset{-1}{\overset{1}{\rightleftarrows}} \circ \underset{-1}{\overset{1}{\rightleftarrows}} \circ \; t$

Residual graph G_f, f feasible flow in G.

Capacities $\quad c_f(u,v) = c(u,v) - f(u,v)$

augmenting path: path from s to t whose edges have strictly positive residual capacity.

Def. An s-t cut in flow network $G = (V, s, t, c)$ is a partition of V into S, T with $s \in S$, $t \in T$.

For vertex sets Q, R let

$$f(Q,R) = \sum_{u \in Q} \sum_{v \in R} f(u,v) \qquad \text{net flow } Q \to R$$

$$c(Q,R) = \sum_{u \in Q} \sum_{v \in R} c(u,v) \qquad \text{aggregate capacity } Q \to R$$

Observe: (a) R_1, R_2 disjoint $\Rightarrow$ $f(Q, R_1 \cup R_2) = f(Q, R_1) + f(Q, R_2)$

(b) $f(Q, Q) = 0 \quad \forall Q \subseteq V$

... by skew-symmetry.

Lemma. If f is any flow and S, T is any s-t cut,

$$f(S,T) = \mathrm{val}(f).$$

If f feasible,

$$\mathrm{val}(f) \leq c(S,T)$$

and equality holds iff and only if $f(u,v) = c(u,v)$ for all $u \in S$, $v \in T$. ("S,T is saturated by f.")

Proof. By properties (a), (b) above,

$$f(S,T) = f(S,T) + f(S,S) = f(S, T \cup S) = f(S,V)$$

$$= \sum_{u \in S} \sum_{v \in V} f(u,v)$$

inner sum equals zero except when $u = s$.

$$= \sum_{v \in V} f(s,v) = \mathrm{val}(f).$$

Inequality $\mathrm{val}(f) \leq c(S,T)$ follows from

$$\forall\, u,v \qquad f(u,v) \leq c(u,v)$$

at least one of these is strict <

for $u \in S, v \in T$

$$\underbrace{f(S,T)}_{=\mathrm{val}(f)} = \sum_{u \in S} \sum_{v \in T} f(u,v) \leq \sum_{u \in S} \sum_{v \in T} c(u,v) = c(S,T)$$

this is strict <,

Theorem (Max-Flow Min-cut) For a flow network G and a feasible flow f, TFAE:

(i) f is a maximum flow

(ii) there is no augmenting path in G_f

(iii) there exists a s-t cut with $c(S,T) = \mathrm{val}(f)$

(iv) there exists a minimum s-t cut with $c(S,T) = \mathrm{val}(f)$.

textbook version

Proof. First (iv) $\Rightarrow$ (iii) obvious. To prove (iii) $\Rightarrow$ (iv) assume f, S, T satisfy (iii) and assume

S^*, T^* is any s-t cut of minimum capacity.

$$c(S^*, T^*) \leq c(S,T) \qquad \text{(def of } S^*, T^*\text{)}$$

lemma above

$$c(S,T) = \text{val}(f) \leq c(S^*,T^*)$$

Hence $c(S^*,T^*) = c(S,T)$ are equal, so S,T is a minimum s-T cut satisfying $\text{val}(f) = c(S,T)$ as required by (iv).

For (i) $\Rightarrow$ (ii) we prove $(\neg ii) \Rightarrow (\neg i)$.
If G_f has aug path P, let

$$\delta(P) = \min \{ c(u,v) - f(u,v) \mid (u,v) \text{ an edge of } P \}$$

Then $f + \delta(P) \cdot f^P$ is also a feasible flow, its value is $\text{val}(f) + \delta(P) > \text{val}(f)$, so f is not a max flow.

For (ii) $\Rightarrow$ (iii): define an <u>augmenting walk</u> to be a sequence $s = u_0, u_1, u_2, \ldots, u_k$ of vertices, s.t. residual cap > 0 for all (u_i, u_{i+1}), $0 \leq i < k$.
$c(u_i, u_{i+1}) - f(u_i, u_{i+1})$

Let $S = \{ u \mid \exists \text{ an augmenting walk ending at } u \}$

$T = V \setminus S$

Note $s \in S$ because (s) is augmenting walk
$t \in T$ because $\nexists$ augmenting path in G_f.

Every (u,v) with $u \in S$, $v \in T$ has zero residual capacity. This is because
$\exists$ augmenting walk $s = u_0, u_1, \ldots, u_k = u$
but $u_0, u_1, \ldots, u_k, u_{k+1} = v$ is <u>not</u> an augmenting walk. $\Rightarrow$ (u,v) has ≤ 0 residual capacity. $\therefore = 0$

We have an s-t cut which is saturated by f, so $c(S,T) = \text{val}(f)$ by lemma above.

Lastly, for (iii) $\Rightarrow$ (i):

If f is feasible flow,
S,T is s-t cut
and $val(f) = c(S,T)$ } condition (iii)

then for any feasible flow f^*,

$$val(f^*) \underset{\text{by Lemma}}{\leq} c(S,T) = val(f)$$

$\therefore$ $val(f)$ is the maximum value of a feasible flow in G.