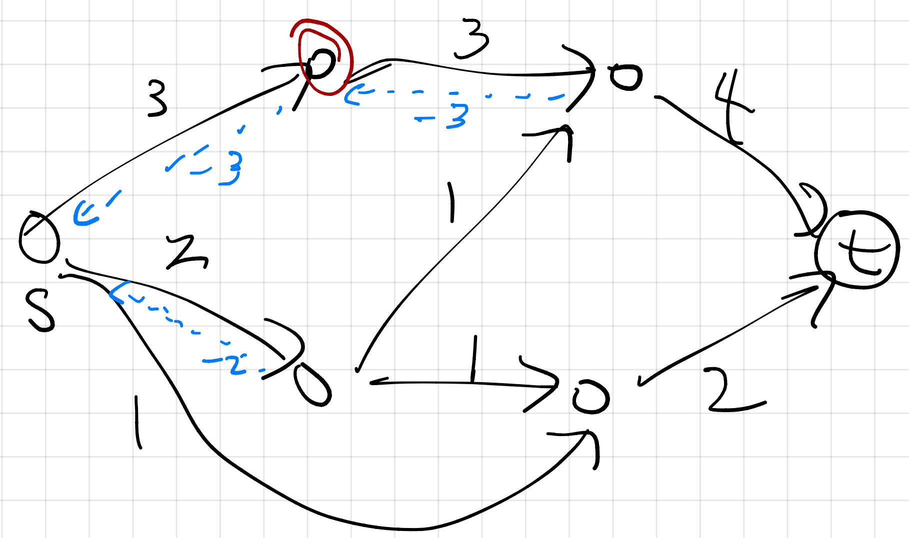
22 Sep 2023

Flows

Announcements:

1. Lecture notes to be posted in a few days.
2. NO CLASS MON 9/25.

What does a flow look like?



Def. A <u>flow</u> with vertex set V, source and sink $s,t \in V$ is a function $f: V \times V \to \mathbb{R}$ s.t.

- skew symmetry: $f(u,v) + f(v,u) = 0 \quad \forall u,v$
- flow conservation: $\sum_{v \in V} f(u,v) = 0 \quad \forall u \neq s,t$

The <u>value</u> of a flow, $\mathrm{val}(f)$, is

$$\mathrm{val}(f) = \sum_{v \in V} f(s,v).$$

A <u>flow network</u> (V, s, t, c) is a vertex set V, source s, sink t, and capacity function

$$c: V \times V \to [0, \infty]$$

Flow f is <u>feasible</u> if $f(u,v) \leq c(u,v) \quad \forall u,v \in V$.

The maximum s-t flow problem is:
given (V, s, t, c) find a feasible flow f maximizing $\mathrm{val}(f)$. (If one exists, i.e. $\sup\{\mathrm{val}(f)\} < \infty$.)

A flow is a "weighted sum of paths and cycles".

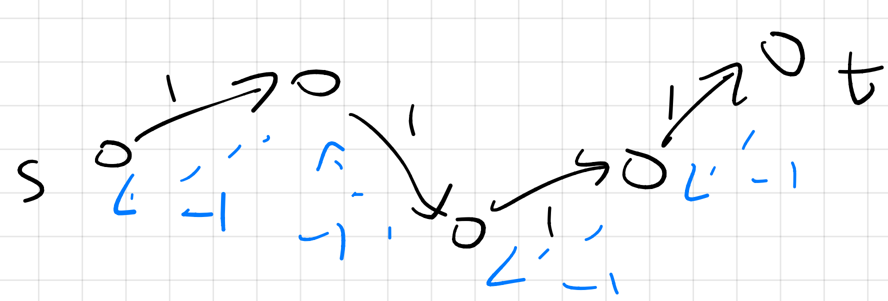
<u>Def.</u> Suppose $P = v_0, v_1, \ldots, v_k$ is a sequence in V and either

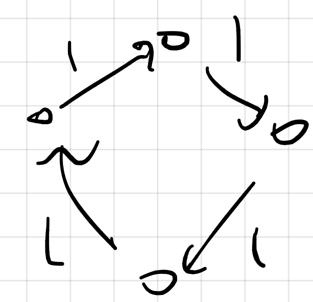
(i) P is a simple path with endpoints s,t:
$\{v_0, v_k\} = \{s,t\}$ and $v_0, \ldots, v_k$ distinct

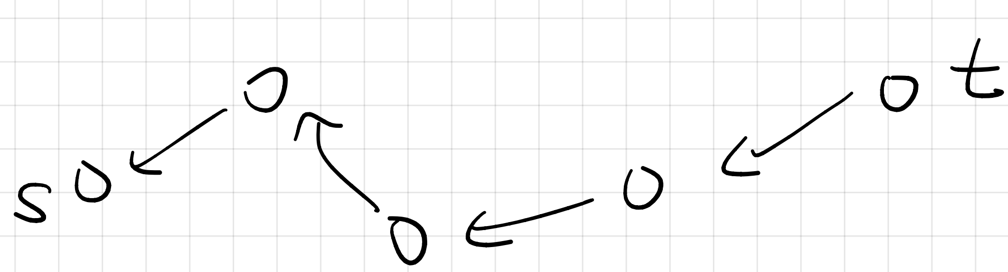
(ii) P is a simple cycle
$v_0 = v_k$ and $v_1, \ldots, v_k$ distinct.

Then the <u>elementary flow</u> assoc to P is

$$f^P(u,v) = \begin{cases} 1 & \text{if } \exists i \quad u = v_i, \ v = v_{i+1} \\ -1 & \text{if } \exists i \quad v = v_i, \ u = v_{i+1} \\ 0 & \text{otherwise.} \end{cases}$$







<u>Lemma.</u> Every flow is a non-negative weighted sum of elementary flows, $f = \sum_P w_P f^P$.
And $\mathrm{val}(f) = \left(\sum_{P\ s \to t} w_P \right) - \left(\sum_{P\ t \to s} w_P \right)$ in any such decomposition.

Def. If f is a feasible flow in network $G = (V, s, t, c)$ the <u>residual</u> <u>network</u> of f is $G_f = (V, s, t, c - f)$.

Lem. If f is a feasible flow in $G = (V, s, t, c)$ there is a bijection

$$\left\{ \begin{array}{c} \text{feasible flows} \\ f' \text{ in } G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{feasible flows} \\ h \text{ in } G_f \end{array} \right\}$$

$$f' \longmapsto f' - f$$

$$h + f \longleftarrow h$$

$\forall u, v \quad h(u,v) + f(u,v) \leq c(u,v)$ $\qquad$ $\forall u, v \quad h(u,v) \leq c(u,v) - f(u,v)$

This bijection is value-preserving up to an additive shift of $\pm \operatorname{val}(f)$.

Cor. f is a max flow in G if and only if 0 is a max flow in G_f.

If G_f contains a path P from s to t made up of edges with capacity > 0, a.k.a. "augmenting path"

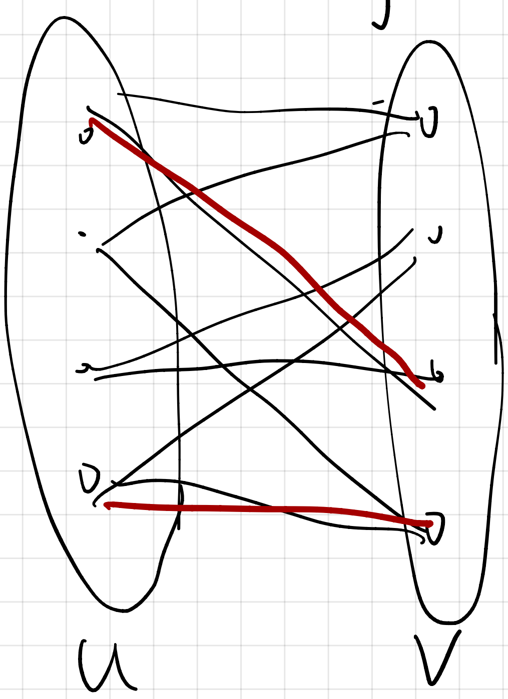
let $\delta = \min \{ c(u,v) - f(u,v) \mid (u,v) \text{ an edge of } P \}$

and observe $\delta \cdot f^P$ is feasible in G_f

$\operatorname{val}(\delta \cdot f^P) = \delta > 0 \Longrightarrow f$ is not a max flow in G.

Connecting max-flow with bipartite matching.

Instance of max bipartite matching



Flow network