8 Sep 2023 — Analysis of RANKING

1. Sample a uniformly random total ordering of L
2. Whenever $v \in R$ arrives, if it has at least one free neighbor, match to the one that comes earliest in this ordering.

$$\begin{aligned} \max \quad & \sum_{(u,v)\in E} x_{uv} \\ \text{s.t.} \quad & \sum_{v} x_{uv} \le 1 \quad \forall u \in L \\ & \sum_{u} x_{uv} \le 1 \quad \forall v \in R \\ & x_{uv} \ge 0 \end{aligned} \qquad \le \qquad \begin{aligned} \min \quad & \sum_{u \in L} y_u + \sum_{v\in R} y_v \\ \text{s.t.} \quad & y_u + y_v \ge 1 \quad \forall (u,v) \in E \\ & y_u, y_v \ge 0 \quad \forall u \in L, v \in R \end{aligned}$$

In terms of duality, here's why GREEDY is a 2-approximation.

As you run GREEDY each time edge (u,v) is selected it "yields \$2 of revenue" which we "reinvest" by putting \$1 on u, \$1 on v.
(Translation: set $y_u = y_v = 1$.)

Why does every edge (u,v) satisfy $y_u + y_v \ge 1$?

Case 1. When v arrived, GREEDY found a match for it.
$\longrightarrow$ it invested \$1 in v, $y_v = 1$.

Case 2. No match for v was available.
$\Longrightarrow$ u was already matched, $y_u = 1$.

$2 \cdot |ALG| =$ total investment $\ge |OPT|$.

Analysis plan for RANKING.

1. When (u,v) is selected we have $\$\left(\frac{e}{e-1}\right) \approx \$1.58$ to divide between y_u and y_v.

2. We'll show how to do it randomly such that $\mathbb{E}[y_u + y_v] \geq 1$ for every edge (u,v).

(whether selected for matching or not.)

Then if we define $\tilde{y}_u = \mathbb{E}[y_u]$, $\tilde{y}_v = \mathbb{E}[y_v]$,

$$\frac{e}{e-1} \cdot \mathbb{E}[ALG] = \mathbb{E}[\text{total investment}]$$

$$= \sum \tilde{y}_u + \sum \tilde{y}_v \geq OPT$$

weak duality

Reinterpretation of RANKING.

1. For all $u \in L$, sample $z_u \in [0,1]$ uniformly at random. (independently for all u)

2. When $v \in R$ arrives, if at least one neighbor is free, match v to free u with smallest z_u.

3. Set
$$y_u = \frac{e}{e-1} \cdot h(z_u)$$
$$y_v = \frac{e}{e-1} \cdot [1 - h(z_u)]$$

$h(z) = e^{z-1}$

Analysis needs to prove $\mathbb{E}[y_u + y_v] \geq 1$ for all $(u,v) \in E$.

Fix one edge (u,v). Fix random z_w $\forall w \neq u$.

Imagine re-running RANKING on $G \setminus \{u\}$.

"Critical value" $z^c := \begin{cases} z_w & \text{if RANKING on } G\setminus\{u\} \text{ matched } v \text{ to } w. \\ 1 & \text{if } v \text{ remained unmatched.} \end{cases}$

Obs 1. If $Z_u < z^c$ then u will be matched by RANKING on G.

Why? Either u was matched before v arrived or its the lowest Z-value among v's free neighbors.

Cor. $\mathbb{E}[y_u] \geq \frac{e}{e-1}\int_0^{z^c} h(z)\, dz$

Obs 2. $y_v \geq \frac{e}{e-1}\left(1 - h(z^c)\right)$ holds at the end of running RANKING.

If v gets matched to u' the ineq says

$$\frac{e}{e-1}\left(1-h(Z_{u'})\right) \geq \frac{e}{e-1}\left(1-h(z^c)\right)$$

$$\Updownarrow$$

$$h(Z_{u'}) \leq h(z^c)$$

$$\Updownarrow$$

$$Z_{u'} \leq z^c.$$

(v chooses from a smaller set of options in $G \setminus \{u\}$)

Obs 1 + Obs 2. $\mathbb{E}[y_u] + \mathbb{E}[y_v] \geq \frac{e}{e-1}\left[\underbrace{\int_0^{z^c} h(z)\, dz + 1 - h(z^c)}\right]$

Our choice of h makes this $\rightarrow$ equal to $1 - \frac{1}{e}$ for every z^c.