CS 6817: Special Topics in Complexity Theory	**Spring 2025**

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1 The Hamming Cube

The Hamming cube is defined as $\{0,1\}^n$, or equivalently $\{\pm 1\}^n$. The Hamming distance between two points over the cube $x, y \in \{0,1\}^n$ is defined as

$$\Delta(x,y) \stackrel{def}{=} |\{i : x_i \neq y_i\}|.$$

The Hamming weight $w \colon \{0,1\}^n \to [0,n]$ of a points $x \in \{0,1\}^n$ is defined as its number of non-zero coordinates, or alternatively—it's distance from 0:

$$w(x) \stackrel{def}{=} \Delta(x,0).$$

Through out this lecture, we define the graph $H_n = (\{0,1\}^n, E)$ over the Hamming Cube, with edges corresponding to vertices with hamming distance 1:

$$(x,y) \in E \iff \Delta(x,y) = 1.$$

2 Influence

We begin with introducing the notion of bit and total *influence* of a Boolean function. Intuitively, the i-th influence of a Boolean function $f \colon \{\pm 1\}^n \to \{\pm 1\}$ would captures the influence the i-th coordinate has over the output, given a random input.

Specifically, for every $x \in \{\pm 1\}^n$ and $i \in [n]$, we define the vector $x^{\oplus i}$ as identical to x except with its i-th coordinate flipped. Then,

Definition 2.1 (Influence)**.** *For every $i \in [n]$, the i-th* Influence *of a Boolean function f is defined as*

$$I_i[f] \stackrel{def}{=} \Pr_{x \sim \{\pm 1^n\}} \left[f(x) \neq f(x^{\oplus i}) \right].$$

Correspondingly, the total influence is defined as the sum of the influences of all bits.

Definition 2.2. *The* Total Influence *is defined as*

$$I[f] \stackrel{def}{=} \sum_{i \in [n]} I_i[f].$$

Is there any bound on the influence of a function? Clearly every bit-influence is bounded as $I_i[f] \in [0,1]$ as it's a probability. Hence, the total influence is bounded between $I[f] \in [0,n]$.

Example 2.1. *The constant function $f = 1$ has no influence at all, namely $I[1] = 0$. The parity function $f(x) = \chi_{[n]}(x)$ has maximum influence $I[f] = n$, since $I_i[\chi_{[n]}] = 1$ (for every index i) as every bit-flip of the input also flips the output.*

A more interesting example is of the *majority* function $Maj_{2k+1}\colon \{\pm 1\}^{2k+1} \to \{\pm 1\}$, which is defined as follows

$$Maj_{2k+1}(x) \overset{def}{=} \begin{cases} 1 & \sum_{i\in[2k+1]} x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1. $I(Maj_{2k+1}) = O(\sqrt{k})$.

Proof. Without loss of generality consider the first coordinate $i = 1$, and observe that

$$I_1[f] = \Pr_{x\sim\{\pm 1\}^n}\left[\sum_{i\in[2,2k+1]} x_i = 0\right] = \frac{\binom{2k}{k}}{2^{2k}},$$

since we need to fix k coordinate with value 1 and the rest as 0. Using Stirling's approximation $\binom{2k}{k} \sim \frac{2^{2k}}{\sqrt{k\pi}}$, we bound the above by

$$\leq \frac{\frac{2^{2k}}{\sqrt{k\pi}}}{2^{2k}} = O(1/\sqrt{k}),$$

and thus

$$I[f] = \sum_{i\in[2k+1]} I_i[f] = (2k+1)\frac{1}{\sqrt{k}} = O(\sqrt{k}).$$

□

Another example is of the $And_n\colon \{\pm 1\}^n \to \{\pm 1\}$ function that is defined as 1 iff $x_1 = \dots x_n = -1$.

Example 2.2. $I[And_n] = \frac{n}{2^{n-1}}$.

Proof. Consider $I_i[And_n]$ for some fixed $i \in [n]$. An input x has influence in coordinate i over the output only if all the other coordinates are -1; a random x satisfies that with probability $2^{-(n-1)}$, and thus

$$I_i[And_n] = 2^{-(n-1)}.$$

Since that's true for every i, the total influence is exactly $n \cdot I_i[And_n]$. □

3 Cuts' perspective

Let us present a different perspective for the Influence. Consider the graph H_n that was defined earlier, and recall that its edges correspond to vertices that differ by 1 coordinate. Define the subset of edges $E_i \subseteq E$ as the ones the differ only on the i-th coordinate, namely $(x, y) \in E_i$ iff $x_i \neq y_i$. Thus,

$$E = \bigcup_{i\in[n]} E_i,$$

where the sets E_i are disjoint.

It follows that $|E_i| = 2^{n-1}$ (for every i), because $(x, y) \in E_i$ if x and y agree on all the coordinates but the i-th one, and there are 2^{n-1} options for such fixation. By the disjoint-ness of E_i we get that

$$|E| = \sum_{i\in[n]} |E_i| = n \cdot 2^{n-1}$$

Let $A_b \overset{def}{=} f^{-1}(b)$ be the set of points corresponds to value $b \in \{\pm 1\}^n$. The Cut between A_1, A_{-1} is the set of all edges that goes from either set to the other:

$$Cut(A_1, A_{-1}) \overset{def}{=} \{(x, y) \in E\colon x \in A_1, y \in A_{-1}\},$$

and correspondingly define the i-th Cut as

$$Cut_i(A_1, A_{-1}) \stackrel{def}{=} Cut(A_1, A_{-1}) \cap E_i.$$

Claim 3.1. *For every $i \in [n]$ it holds that*

$$I_i[f] = \frac{|Cut_i(A_1, A_{-1})|}{|E_i|}.$$

Proof. Enrolling the definition,

$$I_i[f] = \Pr_{x \sim \{\pm 1\}^n}\left[f(x) \neq f(x^{\oplus i})\right],$$

and notice that the "good" x-es, the ones for which $f(x) \neq f(x^{\oplus i})$, namely $(x, x^{\oplus i}) \in Cut_i(A_1, A_{-1})$. There is a probability of $2 \cdot 2^{-n}$ of choosing such x, which is exactly the inverse of the cardinality of $|E_i| = 2^{n-1}$. □

Thus,

Claim 3.2. *It holds that*

$$I[f] = n \cdot \frac{|Cut(A_1, A_{-1})|}{|E|}$$

Proof. Unrolling the definition of total influence,

$$I[f] = \sum_{i \in [n]} I_i[f] = \sum_{i \in [n]} \frac{|Cut_i(A_1, A_{-1})|}{|E_i|}$$

but since all the E_i has the same cardinality $|E_i| = |E_j|$, and thus $|E_1| = |E||/n$, we get that

$$I[f] = \sum_{i \in [n]} \frac{|Cut_i(A_1, A_{-1})|}{|E_1|} = \frac{|Cut(A_1, A_{-1})|}{|E_1|} = n \cdot \frac{|Cut(A_1, A_{-1})|}{|E|}$$

□

4 Sensitivity

Another plausible parameter to measure a Boolean function is it's *sensitivity* at some point x—namely all the points y for which $f(x) \neq f(y)$ and their hamming distance from x is 1:

Definition 4.1. *The* Sensitivity *of f at points x is defined as*

$$S(f, x) \stackrel{def}{=} |\{y \colon f(x) \neq f(y)\} \wedge \Delta(x, y) = 1|.$$

Accordingly, the Average Sensitivity of a function is defined as it sensitivity over a random input:

Definition 4.2. *The* Average Sensitivity *of a Boolean function is defined as*

$$AS(f) = \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n}[S(f, x)].$$

The Average Sensitivity is in fact another perspective for the Total Influence, as they are equal:

Claim 4.1. $AS(f) = I[f]$.

Proof. For every $x \in \{\pm 1\}$ and $i \in [n]$, let $Z_{i,x}$ be an indicator for whether $f(x) \neq x^{\oplus i}$. Thus,

$$S(f, x) = \sum_{i \in [n]} Z_{x,i},$$

and hence by linearity of expectation,

$$AS(f) = \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n} [S(f, x)] = \sum_{i \in [n]} \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n} [Z_{x,i}] = \sum_{i \in [n]} I_i[f] = I[f],$$

where the penultimate equality is by definition of $I_i[f]$. □

The sensitivity of the function itself is defined as the maximum sensitivity over some input x:

Definition 4.3. $S(f) = \max_x S(f, x)$.

For example, $S(AND) = n$ because for $x = (-1, \ldots, -1)$, every bit-flip also flips the AND's value. Other points $y \neq x$, by definitions, have no sensitivity $S(f, y) = 0$.

5 Discrete Derivative

For every $x \in \{\pm 1\}^n$ and $b \in \{\pm 1\}$, define $x^{i \to b}$ identically as x, except the fixture of $x_i = b$.

Definition 5.1. *The i-th* Discrete Derivative *of $f(x)$ is defined as*

$$D_i f(x) \overset{def}{=} \frac{f(x^{i \to 1}) - f(x^{i \to -1})}{2}.$$

Observe that D_i is a linear operator, as for every $\alpha, \beta \in \mathbb{R}$ it holds that

$$D_i(\alpha f) = \alpha D_i(f),$$

and

$$D_i(\alpha f + \beta g) = \alpha D_i(f) + \beta D_i(g).$$

Our first claim relates the derivative's norm to the function's influence.

Claim 5.1. *For every $i \in [n]$,*

$$\|D_i f\|_2^2 = I_i[f].$$

Proof. Observe that $D_i f(x) \in \{-1, 0, 1\}$ as the function's domain is $f(x) \in \{\pm 1\}$.

Thus, by the norm's definition, and the symmetry about choosing $x^{i \to 1}$ and $x^{i \to -1}$,

$$\|D_i f\|_2^2 = \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n} \left[(D_i f(x))^2\right] = \frac{1}{4} \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n} \left[\left(f(x^{i \to 1}) - f(x^{i \to -1})\right)^2\right] = \frac{1}{2} \mathop{\mathbb{E}}_{x \sim \{\pm 1\}^n} \left[\left|f(x^{i \to 1}) - f(x^{i \to -1})\right|\right].$$

Observe that the absolute value does not vanishes only if $f(x^{i \to 1}) \neq f(x^{i \to -1})$, so the above is equal to

$$= \frac{1}{2} 2 \Pr_{x \sim \{\pm 1\}^n} \left[f(x^{i \to 1}) \neq f(x^{i \to -1})\right] = I_i[f],$$

where the factor 2 comes the symmetry of fixing either $x^{i \to 1}$ or $x^{i \to -1}$, and the ultimate equation follows by definition. □