CS 6817: Special Topics in Complexity Theory **Spring 2025**

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1 Introduction

In this section, we often look at functions f as black boxes, allowing us to analyze the input and output but not allowing us access to their formulae / inner workings. We then pick up where we left class last:

Theorem 1.1. *There is a* $(3,1)$*-tester* A *for* $\texttt{LIN} = \{f : \mathbb{F}_2^2 \to \mathbb{F}_2 \mid f \texttt{ is a linear function}\}$.

As discussed in last class, this means that A makes at most 3 queries of f.

- If $f \in \texttt{LIN}$, then A accepts.
- IF $f \notin \texttt{LIN}$, then $\mathbb{P}[A \texttt{ rejects}] \geqslant \lambda \cdot \texttt{dist}(f, \texttt{LIN})$, where $\lambda = 1$.

Definition 1.2. $\texttt{dist}(f, g) := \mathsf{dist}(f, g) = \mathbb{P}[f \neq g]$.

Definition 1.3. $\mathsf{dist}(f, P) := \min_{g \in P}\{\texttt{dist}(f, g)\}$ *for any set of functions* P.

2 The BLR Test

The BLR Test provides us with a test which can be used for the above theorem (1.1), where we will define the test and prove that it meets this condition in this lecture. There are two analogous forms of the BLR Test:

Definition 2.1. *Suppose we have a function* $f : \mathbf{F}_2^n \to \mathbf{F}_2$*. Then, the* BLR *Test is as follows:*

- *First, generate a random* x *and* y*.*
- *Then, accept if and only if* $f(x+y) = f(x) + f(y)$*.*

Likewise, we can define the BLR Test for the other two value basis we've discussed in this course:

Definition 2.2. *The component-wise multiplication operator is defined as* $x \oplus y := \{z \mid z_i = x_i y_i\}$*.*

Definition 2.3. *Suppose we have a function* $f : \{-1,1\}^n \to \{-1,1\}$*. Then, the* BLR *Test is as follows:*

- *First, generate a random* x *and* y*.*
- *Then, accept if and only if* $f(x)f(y)f(x \oplus y) = 1$*.*

Here, we see that for both definitions, we only require 3 calls to f.

We will then try to show that this second definition meets the rejection requirements of the above theorem. Importantly, when our set is $\{0,1\}^n$, $\mathsf{LIN} = \{f \in \mathsf{PARITY}\}$, where PARITY is the set of $\chi_S = \prod_{i \in S} x_i$ that we have discussed in previous lectures.

Suppose that for all $S \in [n]$ f is far from χ_S (namely that f is not equal to any of them). Thus, $\min_{S \subseteq [n]} \{\mathbb{P}[f(x) \neq \chi_s(x)]\} > 0$, so there exists an $\epsilon > 0$ such that $\min_{S \subseteq [n]} \{\mathbb{P}[f(x) \neq \chi_s(x)]\} \geqslant \epsilon$. Equivalently, $\min_{S \subseteq [n]} (\mathtt{dist}(f, \chi_S)) \geqslant \epsilon$.

Previously, we noted that for a random variable X in $\{-1, 1\}$, $\mathbb{E}[]X] = \mathbb{P}[X = 1] - \mathbb{P}[X = -1]$. This gives us the following:

$$\mathbb{E}[f(x)f(y)f(x \oplus y)] = \mathbb{P}[f(x)f(y)f(x \oplus y) = 1] - \mathbb{P}[f(x)f(y)f(x \oplus y) = -1]$$
$$\mathbb{E}[f(x)f(y)f(x \oplus y)] = 2\mathbb{P}[f(x)f(y)f(x \oplus y) = 1] - 1$$
$$\frac{\mathbb{E}[f(x)f(y)f(x \oplus y)] + 1}{2} = \mathbb{P}[f(x)f(y)f(x \oplus y) = 1]$$

As defined previously, $\mathbb{P}[f(x)f(y)f(x \oplus y) = 1] = \mathbb{P}[\mathtt{BLR\ accepts}]$.

Using the Fourier transformation $f(x) = \sum_S \hat{f}(S)\chi_S(x)$, we can then take this expectation and simplify it:

$$\begin{aligned}
\mathbb{E}[f(x)f(y)f(x \oplus y)] &= \mathop{\mathbb{E}}_{x,y}\left[\left(\sum_S \hat{f}(S)\chi_S(x)\right)\left(\sum_T \hat{f}(T)\chi_T(y)\right)\left(\sum_W \hat{f}(W)\chi_W(x \oplus y)\right)\right] \\
&= \sum_{S,T,W} \hat{f}(S)\hat{f}(T)\hat{f}(W) \mathop{\mathbb{E}}_{x,y}[\chi_S(x)\chi_T(y)\chi_W(x \oplus y)] \qquad (1) \\
&= \sum_{S,T,W} \hat{f}(S)\hat{f}(T)\hat{f}(W) \mathop{\mathbb{E}}_x[\chi_S(x)\chi_W(x)] \mathop{\mathbb{E}}_y[]\chi_T(y)\chi_W(y)]
\end{aligned}$$

In this, we are able to first pull out the Fourier-transformed functions, which are not dependent on x and y. Then, because our choice for x and y are independent of each other, we are able to separate their expectations. We can then simplify this further using the following:

Definition 2.4. *For any sets S and T, the symmetric difference is defined as $S \Delta T := \{u \mid u \in S \cup u \notin T \cap u \mid u \notin S \cup u \in T\}$.*

Now, we know that for all sets $S, T \in [n]$, $\chi_S(x)\chi_T(x) = \chi_{S \Delta T}(x)$. Thus, we have the following:

$$\mathbb{E}[f(x)f(y)f(x \oplus y)] = \sum_{S,T,W} \hat{f}(S)\hat{f}(T)\hat{f}(W) \mathop{\mathbb{E}}_x[\chi_{S \Delta W}(x)] \mathop{\mathbb{E}}_y[\chi_{T \Delta W}(y)]$$

In the previous lecture, we showed that $\chi_{S \Delta T}(x) = 1$ when $S = T$ and 0 otherwise. This cancels out most of the terms and yields this result:

$$\mathbb{E}[f(x)f(y)f(x \oplus y)] = \sum_S \widehat{f}(S)^3$$

.

Now, we know that $\sum_S \widehat{f}(S)^3(S) \leqslant \max_S \{\hat{f}(S)\} \sum_S \widehat{f}(S)^2$. Since $\sum_S \widehat{f}(S)^2 = 1$ by Parseval's Theorem, $\sum_S \widehat{f}^{\ 3}(S) \leqslant \max_S \{\hat{f}(S)\}$.

Note, however, that $\hat{f}(S) = 1 - 2\mathbb{P}[f \neq \chi_S]$. By our assumption that $\min_{S \subseteq [n]} \{\mathbb{P}[f(x) \neq \chi_s(x)]\} \geqslant \epsilon$, $\hat{f}(S) \leqslant 1 - 2\epsilon$.

Thus, $\sum_S \hat{f}^{\,3}(S) \leqslant 1 - 2\epsilon$. Plugging this into the above expression, we get $\mathbb{P}\{\texttt{BLR accepts}\} \leqslant 1 - \epsilon$. This means that $\mathbb{P}\{\texttt{BLR rejects}\} \geqslant \epsilon$ which satisfies the requirements of the theorem $\square$.

3 Influence and Noise Probability of Boolean Function

Often, we aim to design Boolean functions with real-world applications. One of these involves dealing with election results, where we seek functions with the following properties:

- **Monotonic**: Define the operator less than or equal to as $x \leqslant y$ if and only if $x_i \leqslant y_i, \forall i$. If $x \leqslant y$, then $f(x) \leqslant f(y)$.
- **Unanimous**: $f(1, 1, \ldots, 1) = 1$ and $f(0, 0, \ldots, 0) = 0$.
- **Symmetric**: For all permutations π, $f(x) = f(\pi(x))$.
- **Odd**: $f(\bar{x}) = \bar{f(x)}$.

We can show that the only function that satisfies these properties is the majority function (`MAJ` on an odd number of bits.

We now introduce the notion of influence or coordinates on Boolean functions, as well as the notion of total influence.

Definition 3.1. *Given $x \in \{-1, 1\}^n$, $x^{\oplus i}$ is equivalent to x with the ith bit flipped.*

Definition 3.2. *Suppose we have a function $f : \{-1, 1\}^n \to \{-1, 1\}$. Then, $I_i(f) = \mathbb{P}_{x \in \{-1,1\}^n} \left[f(x) \neq f\left(x^{\oplus i} \right) \right]$.*

Definition 3.3. *Suppose we have a function $f : \{-1, 1\}^n \to \{-1, 1\}$. Then, the total influence is defined as $I(f) = \sum_{i=1}^n I_i(f)$.*