CS 6210: Homework 5

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Due: November 25, 2024

Policies

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Your solution, including plots and requested output from your code should be typeset and submitted via the Gradescope as a pdf file. Additionally, please submit any code written for the assignment. This can be done by either including it in your solution as an appendix, or uploading it as a zip file to the separate Gradescope assignment.

Question 1:

Here we will prove one nice property of the Rayleigh Quotient $\left(r_A(z) = \frac{z^T A z}{z^T z}\right)$ and use that to prove the convergence rate of Rayleigh iteration. Assume that v is an eigenvector of A with simple eigenvalue λ and z is some vector of unit length such that $\text{dist}(v, z) = \mathcal{O}(\epsilon)$.

(a) Prove that if A is real and symmetric then $|\lambda - r_A(z)| \leq \mathcal{O}(\epsilon^2)$.

(b) Assume that A is a $n \times n$ real and symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Under the assumption that Rayleigh iteration converges (technically it does for almost all starting vectors, i.e., all but a set of measure zero) prove that the convergence is asymptotically cubic to some eigenvalue of A.

(c) Assuming that A is not symmetric (or hermitian) argue that in general we may only expect the Rayleigh quotient r_A to provide an eigenvalue estimate satisfying $|\lambda - r_A(z)| \leq \mathcal{O}(\epsilon)$ when $\text{dist}(v, z) = \mathcal{O}(\epsilon)$.

(Hint: it may be useful as an intermediary to prove that for two one dimensional subspaces represented by unit length vectors v and z $\text{dist}(v, z) = \sqrt{1 - (v^T z)^2}$.)

Question 2:

Implement orthogonal iteration (also sometimes known as simultaneous iteration), you may use a built in eigenvalues solver to find the r eigenvalues of the projection of A into the subspace represented by the current iterate.

For the remainder of this problem let A be an $n \times n$ real matrix with the real block Schur factorization (see HW1 for details)

$$A = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ & T_{22} \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^T$$

where Q_1 is the first r columns of Q, T_{11} is $r \times r$ and the remaining dimensions are as inferred. Furthermore, let A have eigenvalues satisfying $|\lambda_1| \geq \cdots \geq |\lambda_r| > |\lambda_{r+1}| \geq |\lambda_{r+2}| \geq \cdots \geq |\lambda_n|$ and assume that the T_{11} has eigenvalues $\lambda_1, \ldots, \lambda_r$.

(a) For real symmetric A of size 100×100 generate instances of this problem where $\lambda_{r+1} \to \lambda_r$ and show your implementation achieves the desired convergence rate for the invariant subspace of interest, both in terms of iterations and as a function of the eigenvalue gap. Think carefully about how to illustrate this.

(b) Now, assume A is not normal and numerically explore how the convergence behavior of orthogonal iteration changes as a function $\|T_{12}\|_F$ and $\|N\|_F$, where N is the off diagonal part of T (i.e., $T = D + N$ where D is diagonal).What are the implications of your observations? This question is deliberately quite open ended—the goal is explore convergence behavior. So, points will be given for reasonable experiments and discussions and are not tied to any highly specific conclusion.

Question 3:

We will now consider some details related to the convergence of the QR algorithm for non-normal matrices and reiterate a few point from class. Suppose that H is an $n \times n$ square, diagonalizable upper Hessenberg matrix with distinct eigenvalues. Say we run the QR algorithm for k steps with real shifts $\{\mu^{(i)}\}_{i=1}^{k}$, i.e., we let $H^{(1)} = H$ and define the iterates for $i = 1, 2, \ldots$ by (1) computing the QR factorization $(H^{(i)} - \mu^{(i)} I) = Q^{(i)} R^{(i)}$ and (2) forming $H^{(i+1)} = R^{(i)} Q^{(i)} + \mu^{(i)}$. (Note the slight index shift from the notation used in class.)

(a) Assume we are given an $n \times n$ upper Hessenberg matrix H and are running the QR algorithm with the Rayleigh shift $\mu^{(k)} = H^{(k)}_{n,n}$. Now, say that at step k we are computing the QR factorization of $H^{(k)} - \mu^{(k)} I$ using Givens rotations and prior to applying the final necessary rotation we observe the structure

$$G^{(n-2)} \cdots G^{(1)} \left(H^{(k)} - \mu^{(k)} I \right) = \begin{bmatrix} R_{11}^{(k)} & R_{12}^{(k)} & R_{13}^{(k)} \\ 0 & a & b \\ 0 & \epsilon & 0 \end{bmatrix}$$

where $R_{11}^{(k)}$ is $n-2 \times n-2$ and a, b, and ϵ are scalars. Derive an expression for $H^{(k+1)}_{n,n-1}$ where $H^{(k+1)} = R^{(k)} Q^{(k)} + \mu^{(k)} I$ and given some conditions under which we would expect that entry to be $\mathcal{O}(\epsilon^2)$. Notably, this argues that we may sometimes expect quadratic convergence of each eigenvalue in the non-symmetric case.

(b) Prove that

$$(H - \mu^{(k)} I)(H - \mu^{(k-1)} I) \cdots (H - \mu^{(1)} I) = Q^{(1)} \cdots Q^{(k)} R^{(k)} \cdots R^{(1)}.$$

In other words, after k steps of this process we have implicitly computed a QR factorization of some polynomial of H whose roots are the shifts.

Question 4:

Assume we are using the Lanczos process to compute some eigenvalues of a real symmetric matrix A and build the Krylov space starting with vector z_0. At step k we denote the current state of our Lanczos recurrence as

$$AV_k = V_{k+1}\widetilde{T}_k \quad \text{where} \quad \widetilde{T}_k = \begin{bmatrix} T_k \\ \beta_k e_k^T \end{bmatrix}.$$

We denote the eigenvalue/vector pairs of T_k as θ_j and q_j respectively.

(a) Prove that every eigenvalue of T_k is an eigenvalue of a matrix within distance $|\beta_k|$ of A (i.e., for every eigenvalue θ_j of T_k there exists a matrix $\widehat{A}$ such that θ_j is an eigenvalue of $\widehat{A}$ and $\|A - \widehat{A}_j\|_2 \leq |\beta_k|$).

(b) Let $T_k = Q\Theta Q^T$ be the eigen-decomposition of T_k, prove that $\widehat{V} = V_k Q$ and Θ are a solution to

$$\begin{aligned} &\min_{U,D} \|AU - UD\|_F^2 \\ \text{s.t.} \quad & U \in \mathbb{R}^{n\times k},\ D \in \mathbb{R}^{k\times k} \\ & U(:,i) \in \mathcal{K}_k(A, z_0)\ i = 1, 2, \ldots, k \\ & U^T U = I \\ & D_{i,j} = 0\ \forall i \neq j \end{aligned}$$