

CS 6210: HOMEWORK 6

Instructor: Anil Damle

Due: December 4, 2023

POLICIES

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Your solution, including plots and requested output from your code should be typeset and submitted via the Gradescope as a pdf file. Additionally, please submit any code written for the assignment. This can be done by either including it in your solution as an appendix, or uploading it as a zip file to the separate Gradescope assignment.

QUESTION 1:

Say we are given a symmetric matrix A with distinct eigenvalues λ_i (ordered $\lambda_1 > \dots > \lambda_n$) and associated eigenvectors v_i . Consider a k dimensional subspace represented by the span of the matrix $Q \in \mathbb{R}^{n \times k}$ with orthonormal columns and let θ_i and y_i be eigenvalue/vector pairs of $T = Q^T A Q$. If $\theta_i = \lambda_i$ for $i = 1, \dots, k$ what can we say about the relationship between Q and v_1, \dots, v_k ? Does your statement still hold if $\theta_i = \lambda_i$ for k distinct but arbitrary indices?

QUESTION 2 (UNGRADED):

The following question will not be graded and you do not have to submit a solution. Nevertheless, it is a result we saw in class and it would be good to think a little bit about how you would prove a result of this type.

Assume we are using the Lanczos process to compute some eigenvalues of a real symmetric matrix A and build the Krylov space starting with vector z_0 . Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ satisfying $\lambda_1 \geq \dots \geq \lambda_n$ and associated eigenvectors v_1, \dots, v_n .

1. After k steps of the Lanczos process let T_k be the tridiagonal matrix generated by the process, prove that $\theta_1 = \lambda_1(T_k)$ (the largest magnitude eigenvalue of T_k) satisfies

$$\lambda_1 \geq \theta_1 \geq \lambda_1 - (\lambda_1 - \lambda_n) \left(\frac{\tan \phi_1}{c_{k-1}(1 + 2\rho_1)} \right)^2$$

where $\cos(\phi_1) = |z_0^T v_1|$, $\rho_1 = \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_n}$, and $c_{k-1}(x)$ is the Chebyshev polynomial of degree $k-1$.

2. Compare and contrast this convergence result with that for the power method (assuming the eigenvalues are such that the power method would converge to λ_1).