Cache Oblivious Algorithms and Data Structures

Theory and Practice

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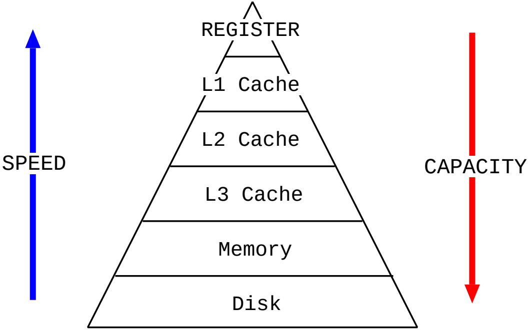
Memory Hierarchy - A Fact of Life

a.k.a. the essence of CS 612

- Multi-level memory hierarchies are omnipresent
- Memory speed $\propto$ $(\text{Distance from processor})^{-1}$
- Good locality is important for achieving high performance



Hardware Parameters

its a jungle out there

- Modern hardware is not uniform - many different parameters
- In homework 1, we used X-RAY to measure
  - CPU speed
  - Instruction Latency/Throughput
  - Number of registers
  - Special Instructions (eg. fma)
  - Cache Stride/Associativity/Capacity/Line-Size/Hit-Latency
- Current programs
  - ignore the parameters - poor performance
  - determine the parameters

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    - automatically, eg. ATLAS generated BLAS libraries

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Is it possible to abstract away this complexity?

- A model that
  - could capture the essence of the hierarchy
  - without knowing its specifics
- Algorithms that are efficient on all hierarchies simultaneously
- and this holy grail is what Cache Oblivious Algorithms aim to attain

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RAM Model

"Introduction to Algorithms" by Cormen et. al.

- The model we use to analyze algorithms in CS 681
- All basic operations take up constant time
- Complexity is the number of operations executed
- Limited practical use
  - Does not take into account the differences of speeds of random access to memory
- Hierarchical Memory Models
  - account for multi-level hierarchies
  - for eg, *Aggarwal et. al.*, '87
  - too complicated for practical use

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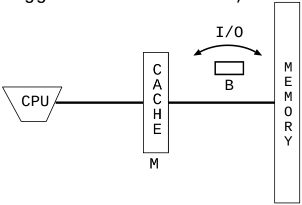
External Memory Model (I/O Model)

a two parameter model

- 2 storage levels
  - Cache
  - Memory
- Complexity - number of transfers between cache and memory

Aggarwal and Vitter, '88



Limitations

- Parameters B and M must be known
- Does not handle multiple memory levels

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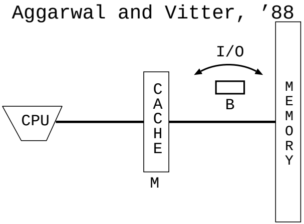

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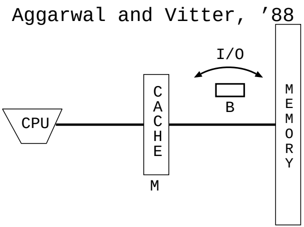

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Cache Oblivious Model (Ideal Cache Model)

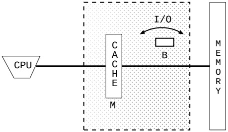
no parameters!

Key Insight

Design algorithms without knowing B and M

- Design
  - Know the existence of the hierarchy
  - Not the parameters

Frigo et. al., '99



Advantages

- Simple and Portable
- Automatically Tuned for hierarchy*
- Efficiency in the asymptotic sense*

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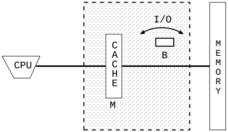

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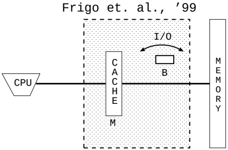

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Assumptions made by the CO model

Frigo et. al.'99

Optimal Cache Replacement

LRU can be used instead, with no *asymptotic* loss in performance Sleator and Tarjan,'85

Full Associativity

Can be simulated in ordinary memory with *constant slowdown* Frigo et. al.,'99

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Random thought

Do these constant factors hurt performance in practice?

So where is this obliviousness used?

In CS612 . . .

- we looked at cache models
- learnt how to transform programs to improve performance

Cache Obliviousness is a tool to build

- Asymptotically efficient algorithms and data structures
- "Programs = Algorithms + Data Structures" - Niklaus Wirth
- Asymptotically efficient programs

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CO Algorithms are hot property

a cottage industry in itself

Their simplicity holds a lot of promise

- Proposed by Frigo et. al. in 1999
- More than 30 papers already Kumar,'03

Existing Algorithms

- Numerical Algorithms: Matrix Mult./Transpose, FFT. . .
- Searching: Van Emde Boas Layout, B-Trees . . .
- Sorting: Funnel Sort, Distribution Sort . . .
- Data Structures: Priority Queues, Ordered File Maintenance . . .
- Other areas include "application-level" problems in computational geometry, graph algorithms, etc.

Main Tool : Divide and Conquer

Divide and Conquer

- Divide the problem recursively
- Solve the trivial problem directly

What's the relation to CO algorithms?

- Trivial problem fits in the cache $\Rightarrow$ good performance
- Results applicable to multi-level hierarchy

Think of CO algorithms as a "catch" phrase

- Divide and Conquer algorithms are CO, for eg. quicksort, mergesort, median selection etc.
- May not achieve optimal performance

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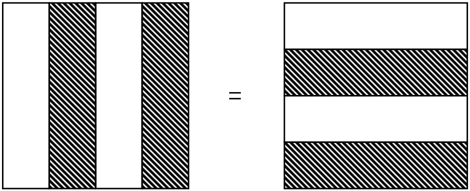
Matrix Transposition : the naive approach

- Transpose *m* X *n* matrix, $B = A^T$

```
for i = 1 to m
 for j = 1 to n
  B(j,i) = A(i,j)
```

Column
Access!!

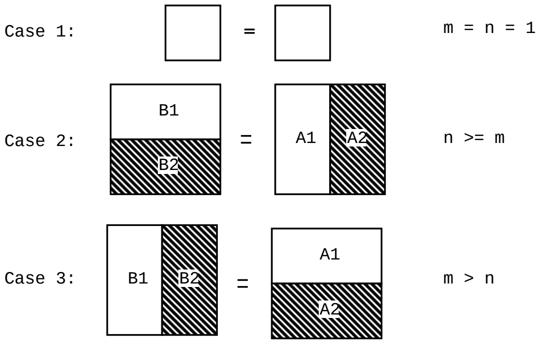


Cache Complexity (# of cache misses)

Q(m,n) = O(mn)

Matrix Transposition : Divide and Conquer

- Transpose m X n matrix, $B = A^T$



Cache Complexity (# of cache misses)

Q(m,n) = O(1 + mn/B), B= Cache Line Size

Algorithms along the same lines

FFT Frigo et. al.'99	$O(1 + n/B(1 + \log_M n))$ Vitter et. al.'94
MMM Frigo et. al.'99	$O(n + n^2/B + n^3/(B\sqrt{M}))$ Hong et. al.'81
Strassen's MMM Frigo et. al.'99	$O(n + n^2/B + n^{\log 7}/(B\sqrt{M}))$ Strassen'69
Median and Selection Demaine'02	$O(1 + n/B)$ Blum et. al.'73
Jacobi's method (2D) Chung et. al.'04	$O((mn)^2/(B\sqrt{M}))$??

Performance in Practice

out of Cinderella's world

- Cache Oblivious Algorithms
  - Good cache performance
  - Poor Execution Time
  - Slower than not so naive algorithms

Mainly due to function call overhead

- Function calls in Matrix Transposition (worst case)
  - log(mn) nodes in the recursion tree
  - (2mn - 1) function calls

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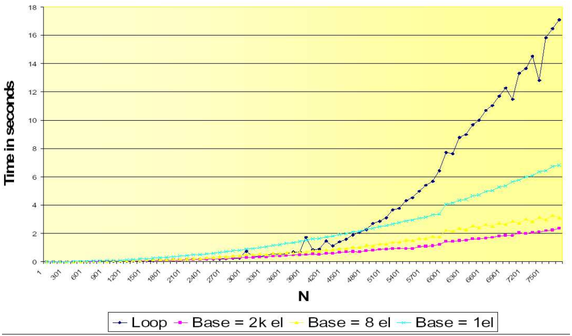
Performance in practice

- from Kumar'03
- Notebook, Windows 2K,512MB RAM, PIII 1GHz, g++ -O3



Need for tuning!!

- Recursion call overhead
- Stop recursion early
    - "... the code is still subject to some tuning, e.g., where to trim the base case of a recursion ..." Demaine,'02

Adaptive Cache Oblivious Algorithms

- Use accurate timing function
- Self-tuning for a good recursion depth

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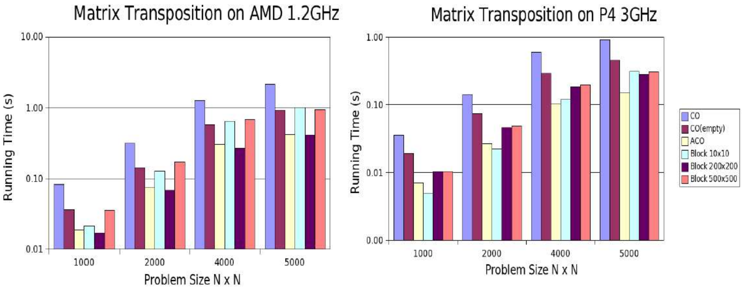
Adaptive Cache Oblivious Algorithms

- Use accurate timing function
- Self-tuning for a good recursion depth

Need for tuning!!

- from Chung'04
- Recursion overhead is significant
- Adaptivity is important



Static Search Tree (Binary Search)

Bender et. al.'00

Static Search Tree

- Fundamental tool in many data structures
- A perfectly balanced binary search tree
- Static : no insertions and deletions

How do we search with few cache misses?

- Optimal bounds
  - Comparisons : O(log N)
  - Memory Transfers : $O(\log_B N)$
- A perfectly balanced binary tree
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- How to minimize the cache misses?

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How to minimize the cache misses?

Prokop'99

Choosing the memory layout

- **Layout** : Mapping of nodes of a tree to memory cells
- Different kinds of layouts
  - In-order
  - Breadth-first
  - Depth-first
  - van Emde Boas

van Emde Boas Layout : Main Idea

Store recursive sub-trees in contiguous memory

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van Emde Boas Layout : Main Idea

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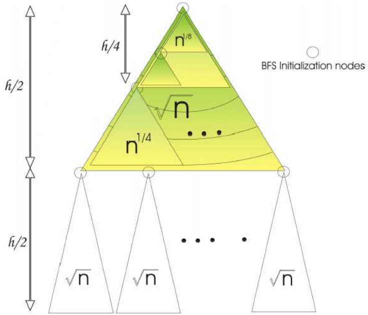
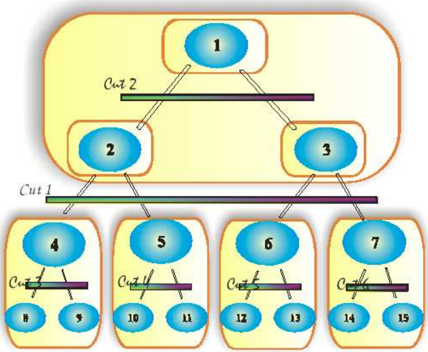
van Emde Boas layout

- Split the tree at the middle level of edges
  - One top recursive subtree
  - $\sim\sqrt{N}$ bottom recursive subtrees : size $\sim\sqrt{N}$
- Recursively layout the top and the bottom subtrees



van Emde Boas layout example

- Tree Height = 4



ACTUAL LAYOUT OF TREE IN MEMORY:

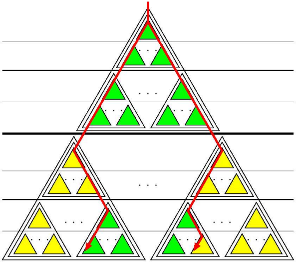
1, 2, 3, 4, 8, 9, 5, 10, 11, 6, 12, 13, 7, 14, 15

How does this help us?

Search complexity

- Recursive subtrees of size at most **B** ⇒ two contiguous blocks
- Two cache misses for each such subtree
- # of cache misses when searching down **log n** levels:

$$(2 \log n) / \log B = 2 \log_B n$$



Is this Divide and Conquer?

- The layout is a kind of divide and conquer
- The algorithm is the usual tree-search algorithm

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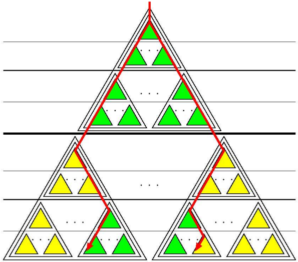

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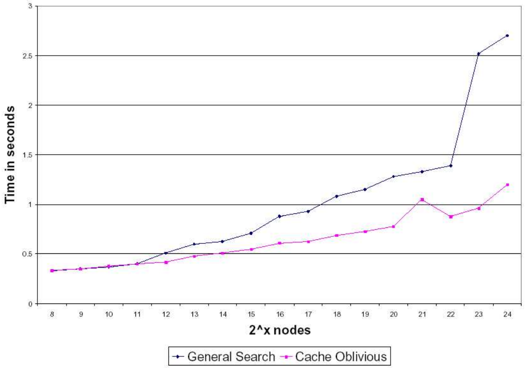
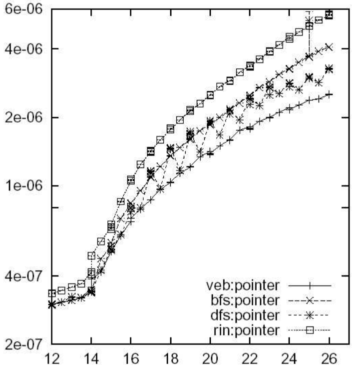
Performance in practice

- from Kumar’03
- Linux/Itanium/2GB/g++ -O3/48 byte nodes

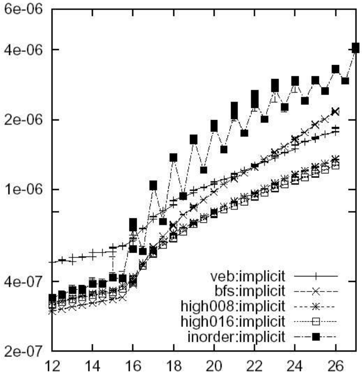


Performance in practice

- Cache Oblivious Search Trees via Binary Trees of Small Height, Brodal et. al.'02
- Linux,Pentium III 1GHz, 256KB cache, 1GB RAM, 4 byte nodes



Another dose of reality!



Take Home Message

One needs to be careful when putting theory into practice

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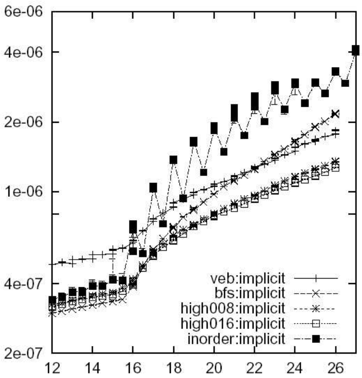

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Some other data structures

Funnels	Prokop'99
Dynamic Search Tree	Bender et. al.'00
Packed Memory Structure	Bender et. al.'00
Priority Queue	Arge et al.'02

Summary

Cache Oblivious Algorithms and Data Structures

- Abstract away the hardware parameters
  - Can handle varying cache specifics and multi-level memory hierarchies while attaining asymptotic efficiency
- A lot of CO algorithms have been developed lately
  - most are generalizations of previous external memory algorithms
  - main techniques : Divide and Conquer, Recursive Layout
- Their innate simplicity holds a lot of promise!!
- A number of issues not addressed by the theoretic model are critical for performance in practical settings