Subcategories

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1 Subcategories

Definition. A Hausdorff space, a.k.a separated space or T_2 space, is a topological space $\langle X, \tau \rangle$ satisfying the "T_2 separation axiom":

$$\forall x, x' \in X.\, x \neq x' \implies \exists O, O' \in \tau.\, x \in O \wedge x' \in O' \wedge O \cap O' = \varnothing$$

Haus is the full subcategory of **Top** comprised of precisely the Hausdorff spaces.

Definition. A Tychonoff space, a.k.a. $T_{3^1/_2}$ space or T_π space or completely T_3 space or completely regular Hausdorff space, is a Hausdorff space $\langle X, \tau \rangle$ satisfying the following "separation axiom":

$$\forall x \in X, O \in \tau.\, x \in O \implies \exists f : \langle X, \tau \rangle \to_{\mathbf{Top}} \mathbb{R}.\, f(x) = 0 \wedge \forall x' \in X \setminus O.\, f(x') = 1$$

Tych is the full subcategory of **Haus** comprised of precisely the Tychonoff spaces.

Definition. **Prost** is the full subcategory of **Rel**(2) of reflexive and transitive relations. **Pos** is the full subcategory of **Prost** of antisymmetric relations.

Definition. Given a preordered set $\langle X, \leq \rangle$, a meet, a.k.a. infimum or greatest lower bound, of an I-indexed collection $\{x_i\}_{i \in I}$ of elements of X is an element x of X satisfying the following properties:

Lower Bound $\forall i \in I.\, x \leq x_i$

Greatest $\forall x' \in X.\, (\forall i \in I.\, x' \leq x_i) \implies x' \leq x$

Given any two meets x and x' of a collection, they are provably equivalent to each other, meaning $x \leq x'$ and $x' \leq x$ holds. As such, we often refer to both as "the" meet of the collection. In fact, if the preorder is actually a partial order, then meets are unique.

An I-indexed meet operator $\bigsqcap$ is a function mapping I-indexed collections to a meet of the input collection. That is, it has the property that $\bigsqcap_{i \in I} x_i$ is always a meet of $\{x_i\}_{i \in I}$. When I has precisely two elements, i.e. the binary case, one often uses the notation $x_1 \sqcap x_2$. When I has no elements, i.e. the nullary case, one often uses the notation $\top$, which is known as a/the top of the preorder. An arbitrary meet operator is a meet operator for every set I or for every set I that is a subset of X.

Definition. Given a preordered set $\langle X, \leq \rangle$, a join, a.k.a. supremum or least upper bound, of an I-indexed collection $\{x_i\}_{i \in I}$ of elements of X is an element x of X satisfying the following properties:

Upper Bound $\forall i \in I.\, x_i \leq x$

Least $\forall x' \in X.\, (\forall i \in I.\, x_i \leq x') \implies x \leq x'$

Given any two joins x and x' of a collection, they are provably equivalent to each other, meaning $x \leq x'$ and $x' \leq x$ holds. As such, we often refer to both as "the" join of the collection. In fact, if the preorder is actually a partial order, then joins are unique.

An I-indexed join operator $\bigsqcup$ is a function mapping I-indexed collections to a join of the input collection. That is, it has the property that $\bigsqcup_{i \in I} x_i$ is always a join of $\{x_i\}_{i \in I}$. When I has precisely two elements, i.e. the binary case, one often uses the notation $x_1 \sqcup x_2$. When I has no elements, i.e. the nullary case, one often uses the notation $\bot$, which is known as a/the bottom of the preorder. An arbitrary join operator is a join operator for every set I or for every set I that is a subset of X.

Definition. A lattice is a partial order with binary meets and joins. A lattice homomorphism is a preorder-preserving function that furthermore preserves binary meets and joins. **Lat** is the subcategory of **Pos** of lattices and lattice homomorphisms.

Definition. A complete lattice is a lattice with arbitrary meets and joins. **JCPos** is the subcategory of **Pos** of complete lattices and arbitrary-join-preserving relation-preserving functions. **CLat** is the subcategory of **Lat** and of **JCPos** of complete lattices and arbitrary-join-preserving and arbitrary-meet-preserving relation-preserving functions.

2 Full and Wide Subcategories

Definition. A semigroup is a set A and an associative binary operator $+ : A \times A \to A$. A semigroup homomorphism is a function that preserves the binary operator. **Sgr** is the category of semigroups and semigroup homomorphisms.

Definition. A full subcategory of a category **C** is a subcategory **S** of **C** that contains all morphisms in **C** between any two objects in **S**.

Example. **Grp** is (isomorphic to) a full subcategory of **Mon** because inverses are unique and all monoid homomorphisms provable preserve inverses. **Mon** is (isomorphic to) a non-full subcategory of **Sgr** because identities are unique but some semigroup homomorphisms do not preserve identities.

Definition. A wide subcategory of a category **C** is a subcategory of **C** that contains all the objects of **C**.

Example. **Set** is (isomorphic to) a wide subcategory of **Rel**.

Example. (**L**)**Met** is a wide subcategory of (**L**)$\mathbf{Met}_u$, which in turn is a wide subcategory of (**L**)$\mathbf{Met}_c$.

3 Isomorphism-Dense/Closed Subcategories and Skeletons

Definition. A (not necessarily full) subcategory **S** of a category **C** is said to be isomorphism-closed whenever every isomorphism in **C** is alwalys contained in **S** if its domain is contained in **S**. Note that one could equivalently define this to use codomain in place of domain.

Example. The following are all (chains of) non-wide isomorphism-closed subcategories:

- **Grp** $\subset$ **Mon** $\subset$ **Sgr**
- **Pos** $\subset$ **Prost** $\subset$ **Rel**(2)
- **CLat** $\subset$ **JCPos** $\subset$ **Pos**
- **CLat** $\subset$ **Lat** $\subset$ **Pos**
- **Met** $\subset$ **LMet**
- **Tych** $\subset$ **Haus** $\subset$ **Top**

Example. For those familiar with linear algebra, **Mat** is (isomorphic to) a skeleton of the category of finite (real-valued) vector spaces and linear functions.