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THE TYPED λ-CALCULUS IS NOT ELEMENTARY RECURSIVE

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Abstract. We prove that the problem of deciding for closed terms t_1, t_2 of the typed λ-calculus whether t_1 β-converts to t_2 is not elementary recursive.

1. Introduction

Historically, the principal interest in the typed λ-calculus is in connection with Gödel's functional ("Dialectica": see Gödel [4]) interpretation of intuitionistic arithmetic. However, since the early sixties interest has shifted to a wide variety of applications in diverse branches of logic, algebra and computer science. For example, in proof-theory (see for example, Tait [20]), in constructive logic (see for example, Lauchli [10]), in the theory of functionals (see for example, Friedman [3]), in cartesian closed categories (see for example, Mann [11]), in automatic theorem proving (see for example, Huet [8]), in the semantics of natural languages (see for example, Montague [14]), and in the semantics of programming languages (see for example, Milner [12]).

In almost all such applications there is a point at which one must ask, for closed terms t_1 and t_2, whether t_1 β-converts to t_2. We shall show that in general this question cannot be answered by a Turing machine in elementary time.

2. Type theory

The language of type theory, Ω, is the language of set-theory where each variable has a natural number type and there are two constants **0**, **1** of type 0. We require that prime formulae be "stratified", i.e., each prime formula has one of the forms $\mathbf{0} \in x^1$, $\mathbf{1} \in x^1$ and $y^n \in z^{n+1}$. Arbitrary formulae are built-up from prime ones by $\neg$, $\wedge$, and $\forall$. The intended interpretation of Ω has **0** denoting 0, **1** denoting 1 and x^n ranging over $\mathcal{D}_n$ where $\mathcal{D}_0 = \{0, 1\}$ and $\mathcal{D}_{n+1} = \text{powerset}\ (\mathcal{D}_n)$. If $A = A(x_1^{n_1} \cdots x_m^{n_m})$ and $\alpha_i \in \mathcal{D}_{n_i}$ for $1 \leq i \leq m$ we write $A[\alpha_1 \cdots \alpha_m]$ for A with $x_i^{n_i}$ denoting α_i.

The problem of deciding whether an arbitrary Ω-sentence is true is recursive. In fact there is a quantifier-elimination for Ω-sentences (see Henkin [6]). Briefly, if one extends the language by adding { , } and defines $x^k =_k y^k \Leftrightarrow \forall z^{k-1}(z^{k-1} \in x^k \leftrightarrow z^{k-1} \in y^k)$ for $k > 0$, each $\alpha \in \mathcal{D}_n$ can be defined by $\{x^{n-1}\colon t_1 =_{n-1} x^{n-1} \vee \cdots \vee t_l =_{n-1} x^{n-1}\}$, where $t_1 \cdots t_l$ define the elements of α and $t^{m-1} \in \{y^m\colon A(y^m)\} \Leftrightarrow_{\text{df}} A(t^{m-1})$, when $n > 0$. Thus $\forall x^n A(x^n) \leftrightarrow A(t_1) \wedge \cdots \wedge A(t_p)$ for $t_1 \cdots t_p$ definitions of the members of $\mathcal{D}_n$.

Proposition 1 (Fischer and Meyer, Statman). *The problem of determining if an arbitrary Ω-sentence is true cannot be solved in elementary time* (*see Meyer* [13, *p.* 479 *no.* 7]).

We shall use the above proposition together with a coding argument to prove our principal result (see below).

Let $V_0 = \emptyset$ and $V_{n+1} = \text{powerset}\ (V_n) \cup V_n$. We note in passing the following:

Corollary (for logicians). *Let $\mathcal{L}$ be the language of set theory supplemented by a constant for each V_n; then the problem of determining if an arbitrary Δ_0-sentence of $\mathcal{L}$ is true cannot be solved in elementary time.*

3. Typed λ-calculus

We consider the typed λ-calculus Λ with a single ground type 0, no constants, only power types (→) and β-conversion. The reader not familiar with the typed λ-calculus should consult Hindley *et al.* [7].

We shall adopt the usual convention of ignoring α-conversion (change of bound variables) deleting type superscripts except where important and omitting parentheses selectively (association to the left). We shall also make use of the substitution prefix [/] both for substituting a term for a variable and for substituting a type for 0.

$\boldsymbol{\emptyset} =_{\text{df}} (0 \to 0) \to (0 \to 0)$ is the type of Λ-numbers. It is easy to verify that the closed (i.e., with no free variables) β-normal terms of type $\boldsymbol{\emptyset}$ are just λxx and for each n,

$$\lambda xy \underbrace{x(\cdots(x}_{n} y)\cdots).$$

Letting

$$\boldsymbol{n} =_{\text{df}} \lambda xy \underbrace{x(\cdots(x}_{n} y)\cdots),$$

if t is a closed term of type $\underbrace{\boldsymbol{\emptyset} \to (\cdots(\boldsymbol{\emptyset}}_{m} \to \boldsymbol{\emptyset})\cdots)$ for each $n_1 \cdots n_m$ there is a unique n such that $t\boldsymbol{n}_1 \cdots \boldsymbol{n}_m \beta\eta$ – conv. $\boldsymbol{n}$. In this way t defines an m-ary number-theoretic function.

An extended polynomial is a polynomial built up from 0, 1, +, ·, sg and $\overline{\text{sg}}$ (see Kleene [9, p. 223, no. 9 and no. 10).

Proposition 2 (Schwichtenberg [16], Statman). *The λ-definable m-ary number theoretic functions are just the extended polynomials.*

In particular, there are closed terms **+**, **·**, **sg** and $\overline{\mathbf{sg}}$ which λ-define resp. +, ·, sg, and $\overline{\text{sg}}$.

There are some very short definitions of very large numbers in Λ. Set $s(0)=1$ and $s(n+1)=2^{s(n)}$ and set $a_1=\mathbf{2}$ and $a_{n+1}=([0\to 0/0]a_n)a_1$; by a computation of Church [2, p. 30] $a_n\beta$—conv. $\boldsymbol{s(n)}$.

The λ-definability of the extended polynomials allows us to code the Boolean operations into Λ. The short definitions of large numbers allow us to iterate λ-definable operations a very large (but fixed) number of times. These are precisely the conditions that permit us to simulate the quantifier-elimination for Ω by β-conversion.

The problem of determining for arbitrary closed terms t_1, t_2 of the same type whether t_1 β-conv. t_2 is decidable. By analyzing the normal form algorithm (see [7, p. 73]) it is easy to see that the problem can be solved in $\mathscr{E}^4$ time (here, $\mathscr{E}^4$ is the 5th level of the Grzegorczyk hierarchy; see Grzegorczyk [5]). Thus with respect to this crude classification our lower bound ($\mathscr{E}^3$ = elementary) is best possible.

4. Translation of Ω into Λ

We define recursively $\mathbf{N}_0=\emptyset$ and $\mathbf{N}_{n+1}=\mathbf{N}_n\to\emptyset$. The following definitions are central to what follows.

(1) $e_0 =_{\text{df}} \lambda xy\,\mathbf{+}(\mathbf{\cdot}(\mathbf{sg}\,x)(\overline{\mathbf{sg}}\,y))(\mathbf{\cdot}(\mathbf{sg}\,y)(\overline{\mathbf{sg}}\,x))$.

For all n, m, $(e_0\,\boldsymbol{nm})\beta$-conv. $\boldsymbol{0} \Leftrightarrow n=0=m$ or $0<n$, m, and $(e\,\boldsymbol{nm})$ β-conv. $\boldsymbol{0}$ or $\boldsymbol{1}$.
e_0 has type $\boldsymbol{\emptyset}\to(\boldsymbol{\emptyset}\to\boldsymbol{\emptyset})$.

(2) $\forall_0 =_{\text{df}} \lambda h\,\mathbf{+}(h\boldsymbol{0})(h\boldsymbol{1})$,
$\forall_0$ has type $\mathbf{N}_1\to\boldsymbol{\emptyset}$.

(3) $C =_{\text{df}} \lambda g\,\mathbf{+}(g(\lambda x\boldsymbol{1}))(g(\lambda xx))$,
C has type $\mathbf{N}_2\to\boldsymbol{\emptyset}$.

(4) $p_{n+1}(x,z) =_{\text{df}} C(\lambda f(\forall_n(\lambda w(z(\lambda y\cdot(f(e_nwy);(xy)))))))$.

Here x has type $\mathbf{N}_n\to\boldsymbol{\emptyset}$, y has type $\mathbf{N}_n$, w has type $\mathbf{N}_n$, z has type $\mathbf{N}_n\to\boldsymbol{\emptyset}$, and f has type $\tilde{\boldsymbol{\emptyset}}\to\boldsymbol{\emptyset}$. We have $p_{n+1}(x,z)\beta$-conv. $\mathbf{+}(\forall_n(\lambda w(z(\lambda y\cdot(\lambda x\boldsymbol{1})(e_nwy)(xy)))))(\forall_n(\lambda w(z(\lambda y\cdot((\lambda xx)e_nwy)(xy)))))$. "$C$" stands for "choice (for f)". "p_{n+1}" stands for "prime constituent for building definitions of type $n+1$ objects".

(5) $e_{n+1} =_{\text{df}} \lambda xy\forall_n(\lambda z(e_0(xz)(yz)))$,
e_{n+1}has type $\mathbf{N}_{n+1}\to(\mathbf{N}_{n+1}\to\boldsymbol{\emptyset})$.

(6) $\forall_{n+1} =_{df} \lambda y((([\mathbf{N}_{n+2}/0]a_{n+1})(\lambda zxp_{n+1}(x,z))y)\lambda w\mathbf{1})$,
$\forall_{n+1}$ has type $\mathbf{N}_{n+2} \to \boldsymbol{\emptyset}$.
We now define the translation *:

$$\boldsymbol{0}^* = \boldsymbol{0}$$
$$\boldsymbol{1}^* = \boldsymbol{1}$$
$$(x^n)^* = x^{\mathbf{N}_n}$$
$$(t_1 \in t_2)^* = \mathrm{sg}(t_2^* t_1^*)$$
$$(A \wedge B)^* = \mathrm{sg}(+A^*B^*)$$
$$(\neg A)^* = \overline{\mathrm{sg}}A^*$$
$$(\forall x^n A)^* = \mathrm{sg}(\forall_n \lambda x^{\mathbf{N}_n} A^*).$$

We shall show that for Ω-sentences A, A is true $\Leftrightarrow A^*\beta$-conv. $\boldsymbol{0}$ and $\neg A$ is true $\Leftrightarrow A^*\beta$-conv. $\boldsymbol{1}$. The key idea is that the β-reductions of $\forall_n$ simulate the quantifier-elimination for Ω-sentences. Here $+$ plays the role of $\wedge$ so $\cdot$ plays the role of $\vee$. In addition, e_n plays the role of equality between type n objects. This motivates the definitions below.

5. Verification that the translation is correct

We define the notion of a definition of an object of type n as follows.
(a) $\mathrm{def}^0(0) = \{\boldsymbol{0}\}$,
(b) $\mathrm{def}^0(1) = \{\boldsymbol{1}\}$,
(c) if $\alpha \in \mathscr{D}_{n+1}$, then $\mathrm{def}^{n+1}(\alpha) = \{\lambda y \cdot r_1(\cdots(\cdot\, r_{s(n+1)}((\lambda w\boldsymbol{1})y))\cdots)$: y, w have type $\mathbf{N}_n$, $r_i = \boldsymbol{1}$ or $r_i = e_n t y$ for $t \in \mathrm{def}^n(\beta)$ and $\beta \in \alpha$, for each $\beta \in \alpha$ for some $t \in \mathrm{def}^n(\beta)$ there is some i s.t. $r_i = e_n t y\}$. We set $\mathrm{def}_n = \bigcup_{\alpha \in \mathscr{D}_n} \mathrm{def}^n(\alpha)$.

Below we define sets N_n, orders $\underset{n}{\prec}$ and functions $d_n : N_n \to \mathrm{def}_n$. The members of N_n code various processes of constructing members of def_n and for $\eta \in N_n$, $d_n(\eta)$ is the member of def_n constructed by the process coded by η. The order $\underset{n}{\prec}$ describes a fixed process for generating the processes coded by members of N_n. First some set theoretic preliminaries.

If X and Y are sets then $X \otimes Y = \{(x,y) : x \in X \text{ and } y \in Y\}$ and ${}^XY = \{\eta : \eta : X \to Y\}$. $\pi_1 : X \otimes Y \to X$ is defined by $\pi_1(x,y) = x$ and $\pi_2 : X \otimes Y \to Y$ is defined by $\pi_2(x,y) = y$. If ρ is an ordering of X and γ an ordering of Y, then $\delta = \rho \otimes \gamma$ is the ordering of $Z = X \otimes Y$ defined by $z_1 \delta z_2$ if $\pi_2 z_1 \gamma \pi_2 z_2$ or $\pi_2 z_1 = \pi_2 z_2$ and $\pi_1 z_1 \rho \pi_1 z_2$. $[1,n] = \{k : 1 \le k \le n\}$. ρ^n is the ordering of ${}^{[1,n]}X$ defined by $\eta_1 \rho^n \eta_2$ if $\eta_1 \ne \eta_2$ and for $m = \max\{k : 1 \le k \le n \text{ and } \eta_1(k) \ne \eta_2(k)\}$, $\eta_1(m) \rho \eta_2(m)$.

Define for all n and $1 \le m \le s(n)$, N_n^m as follows; $N_0^1 = \{0,1\}$ and $N_{n+1}^m = [1,m](N_n^{s(n)} \otimes N_0^1)$. Set $N_n = N_n^{s(n)}$ and define $\underset{n}{\overset{m}{\prec}}$ by $\underset{0}{\overset{1}{\prec}}$ is the natural order on N_0^1, $\underset{n+1}{\overset{m}{\prec}} = (\underset{n}{\overset{s(n)}{\prec}} \otimes \underset{0}{\overset{1}{\prec}})^m$. Set $\underset{n}{\prec} = \underset{n}{\overset{s(n)}{\prec}}$.

Define d_n^m or N_n^m as follows: $d_0^1(0)=\boldsymbol{0}$, $d_0^1(1)=\boldsymbol{1}$ and for $\eta\in N_{n+1}^m$, $d_{n+1}^m(\eta)=\lambda y\cdot r_1(\cdots(\cdot\, r_m(xy))\cdots)$ where $r_i=\boldsymbol{1}$ if $\pi_2\eta(i)=0$ and $r_i=e_n([\lambda z\boldsymbol{1}/x]d_n^{s(n)}\,(\pi_1\eta(i)))y$ if $\pi_2\eta(i)=1$. Set $d_n=[\lambda z\boldsymbol{1}/x]d_n^{s(n)}$.

Now suppose that X is a set of occurrences of terms of type σ ordered by ρ, $|X|$ is a power of 2 and $X=X_1\cup X_2$ is a partition of X with $|X_1|=|X_2|$ and $t_1\in X_1$ and $t_2\in X_2\Rightarrow t_1\rho t_2$. Let z be a variable of type $\sigma\to\boldsymbol{\emptyset}$; we define the term $\sum_{t\in X} zt$ recursively by

$$\sum_{t\in X} zt=+\Big(\sum_{t_1\in X_1} zt_1\Big)\Big(\sum_{t_2\in X_2} zt_2\Big).$$

We shall prove the

Proposition 3. $\forall_n\beta$-conv. $\lambda y\sum_{\eta\in N_n} yd_n(\eta)$.

Think of

$$\sum_{\eta\in N_{n+1}} zd_{n+1}^m(\eta)$$

as a symmetric binary tree (branching upwards) with a member of N_{n+1}^m at each leaf. The order of the members from left to right is $\underset{n+1}{\overset{m}{\prec}}$. If we think of a member of N_{n+1}^m as a sequence of pairs then a member of N_{n+1}^{m+1} can be obtained by adding a member of N_{n+1}^1 at the end. Moreover if $\xi\in N_{n+1}^m$ and $\eta\in N_{n+1}^1$, then $[d_{n+1}^1(\eta)/x]d_{n+1}^m(\xi)\beta$-conv. $d_{n+1}^{m+1}(\widehat{\xi\eta})$. In addition if $\xi_1,\xi_2\in N_{n+1}^m$ and $\eta_1,\eta_2\in N_{n+1}^1$, then $\widehat{\xi_1\eta_1}\underset{n+1}{\overset{m+1}{\prec}}\widehat{\xi_2\eta_2}\Leftrightarrow\eta_1\underset{n+1}{\overset{1}{\prec}}\eta_2$ or $\eta_1=\eta_2$ and $\xi_1\underset{n+1}{\overset{m}{\prec}}\xi_2$. From these remarks it is easy to see the

Fact. $\displaystyle\sum_{\eta\in N_{n+1}}\Big(\sum_{\xi\in N_{n+1}^k} z[d_{n+1}^1(\eta)/x]d_n^m(\xi)\Big)\beta\text{-conv.}\sum_{\eta\in N_{n+1}^{k+1}} zd_{n+1}^{k+1}(\eta).$

The members of N_1^1 are $(0,0)(1,0)(0,1)(1,1)$ in the $\underset{1}{\overset{1}{\prec}}$ ordering. We have $\lambda zxp_1(x,z)\beta$-conv. $\lambda zxC\lambda f(+(z(\lambda y\cdot f(e_0\boldsymbol{0}y)(xy)))(z(\lambda y\cdot f(e_0\boldsymbol{1}y)(xy))))$ β-conv. $\lambda zx+(+(z(\lambda y\cdot\boldsymbol{1}(xy)))(z(\lambda y\cdot\boldsymbol{1}(xy))))(+(z(\lambda y\cdot(e_0\boldsymbol{0}y)(xy))))(z(\lambda y\cdot(e_0\boldsymbol{1}y)\cdot(xy))))$. The last term is $\lambda zx\sum_{\eta\in N_1^1} zd_1^1(\eta)$ since $d_1^1((0,0))=\boldsymbol{1}$, $d_1^1((1,0))=\boldsymbol{1}$, $d_1^1((0,1))=e_0\boldsymbol{0}y$ and $d_1^1((1,1))=e_0\boldsymbol{1}y$. More generally we have the

Lemma. *For* $1\leq m\leq s(n+1)([\mathbf{N}_{n+2}/0]\boldsymbol{m})\lambda zx\rho_{n+1}(x,z)\beta$-conv.

$$\lambda zx\sum_{\eta\in N_{n+1}^m} zd_{n+1}^m(\eta).$$

Proof. By induction on (n,m) ordered lexicographically.

Basis: $n=0$.

Case: $m=1$. $([\mathbf{N}_2/0]\mathbf{1})\lambda zxp_1(x,z)\beta$-conv. $\lambda zxp_1(x,z)$ so by the above computation $([\mathbf{N}_2/0]\mathbf{1})\lambda zxp_1(x,z)\beta$-conv. $\lambda zx\sum_{\eta\in N_1^1} zd_1^1(\eta)$.

Case: $m=2$. $([\mathbf{N}_2/0]\mathbf{2})\lambda zxp_1(x,z)\beta$-conv. $\lambda w\lambda zxp_1(x,z)(\lambda zxp_1(x,z)w)\beta$-conv. $\lambda xw\sum_{\eta\in N_1^1}(\lambda yp_1(y,w))d_1^1(\eta)$ by case $m=1$ β-conv.

$$\lambda zx\sum_{\eta\in N_1^1}\Bigl(\sum_{\xi\in N_1^1} z[d_1^1(\eta)/x]d_1^1(\xi)\Bigr)\beta\text{-conv. }\lambda zx\sum_{\eta\in N_1^2} zd_1^2(\eta)\text{ by the fact.}$$

Induction step: $n>0$.

Case: $m=1$. $([\mathbf{N}_{n+2}/0]\mathbf{1})\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda zxC\lambda f\sum_{\eta\in N_n}(\lambda wz(\lambda y\cdot(f(e_nwy))(xy)))d_n(\eta)$ by induction hypothesis β-conv.

$$\lambda zx+\Bigl(\sum_{\eta\in N_n} z(\lambda y\cdot\mathbf{1}(xy))\Bigr)\Bigl(\sum_{\eta\in N_n} z(\lambda y\cdot(e_nd_n(\eta)y)(xy))\Bigr)=\lambda zx\sum_{\eta\in N_{n+1}^1} zd_{n+1}^1(\eta).$$

Case: $m=k+1$. $([\mathbf{N}_{n+2}/0]\boldsymbol{m})\lambda zxp_{n+1}(x,z)\beta$-conv. $\lambda w_1\lambda zxp_{n+1}(x,z)$ $([\mathbf{N}_{n+2}/0]\boldsymbol{k})\lambda zxp_{n+1}(x,z)w_1)\beta$-conv. $\lambda w_1\lambda zxp_{n+1}(x,z)(\lambda x\sum_{\eta\in N_{n+1}^k} w_1d_{n+1}^k(\eta))$ by induction hypothesis β-conv.

$$\lambda zw_2+\Bigl(\sum_{\eta\in N_{n+1}^1}\Bigl(\lambda x\sum_{\xi\in N_{n+1}^k} zd_{n+1}^k(\xi)\Bigr)(\lambda y\cdot\mathbf{1}(w_2y))\Bigr)$$
$$\Bigl(\sum_{\eta\in N_{n+1}^1}\Bigl(\lambda x\sum_{\xi\in N_{n+1}^k} zd_{n+1}^k(\xi)\Bigr)(\lambda y\cdot(e_nd_{n+1}^1(\eta)y)(w_2y))\Bigr)$$

by case $m=1$ β-conv. $\lambda zx\sum_{\eta\in N_{n+1}^m} zd_{n+1}^m(\eta)$ by the fact.

Proof of Proposition 3.

$$\mathbf{V}_{n+1}\beta\text{-conv. }\lambda y\Bigl(\lambda zx\sum_{\eta\in N_{n+1}} zd_{n+1}^{s(n+1)}(\eta)\Bigr)y\lambda w\mathbf{1}$$

by the

$$\text{lemma }\beta\text{-conv. }\lambda y\sum_{\eta\in N_{n+1}} y[\lambda w\mathbf{1}/x]d_{n+1}^{s(n+1)}(\eta)=\lambda y\sum_{\eta\in N_{n+1}} yd_{n+1}(\eta).$$

The proposiition would be useless without the following easy

Observation. *If $\eta\in N_n$, then $d_n(\eta)\in\mathrm{def}^n$ and for each $\alpha\in\mathcal{D}_n$ there is an $\eta\in N_n$ such that $d_n(\eta)\in\mathrm{def}^n(\alpha)$.*

The members of def_1 are

$$\lambda y\cdot\mathbf{1}(\cdot\mathbf{1}((\lambda w1)y)),\ \lambda y\cdot\mathbf{1}(\cdot(e_0\mathbf{0}y)(\lambda w1)y)),$$
$$\lambda y\cdot\mathbf{1}(\cdot(e_0\mathbf{1}y)((\lambda w1)y)),\ \lambda y\cdot(e_0\mathbf{0}y)(\cdot\mathbf{1}((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{1}y)(\cdot\mathbf{1}((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{0}y)(\cdot(e_0\mathbf{0}y)((\lambda w1)y)),$$
$$\lambda y\cdot(e_0\mathbf{0}y)(\cdot(e_0\mathbf{1}y)((\lambda w1)y)),\ \lambda y\cdot(e_0\mathbf{1}y)(\cdot(e_0\mathbf{0}y)((\lambda w1)y))$$

and

$$\lambda y \cdot (e_0 \mathbf{1} y)(\cdot (e_0 \mathbf{1} y)((\lambda w 1)y))$$

so if $\gamma \in \mathcal{D}_1\alpha, \beta \in \mathcal{D}_0 t_1 \in \text{def}^0(\beta) t_2 \in \text{def}^0(\alpha)$ and $t_3 \in \text{def}^1(\gamma)$, then $\beta \in \gamma \Leftrightarrow (t_3 t_1 \beta$-conv. $\mathbf{0})$ and $\alpha = \beta \Leftrightarrow (e_0 t_1 t_2 \beta$-conv. $\mathbf{0})$. More generally we have the

Proposition 4. *Suppose* $\alpha, \beta \in \mathcal{D}_n$, $\gamma \in \mathcal{D}_{n+1}$, $t_1 \in \text{def}^n(\beta)$, $t_2 \in \text{def}^n(\alpha)$, *and* $t_3 \in \text{def}^{n+1}(\gamma)$, *then*

(a) $\beta = \alpha \Leftrightarrow (e_n t_1 t_2 \beta$-conv. $\mathbf{0})$, *and*

(b) $\beta \in \gamma \Leftrightarrow (t_3 t_1 \beta$-conv. $\mathbf{0})$.

Proof. By induction on n.

Basis: $n = 0$. This is the preceding remark.

Induction step: $n = m + 1$.

(a) We have $e_n t_1 t_2 \beta$-conv. $\sum_{\eta \in N_m} e_0(t_1 d_m(\eta))(t_2 d_m(\eta))$ by the previous proposition. If $\beta = \alpha$ and $d_m(\eta) \in \text{def}^n(\beta)$ by hyp. ind. on (b) $e_0(t_1 d_m(\eta))(t_2(d_n(\eta)))$ β-conv. $\mathbf{0}$ and if $d_m(\eta) \notin \text{def}^n(\beta)$ by hyp. ind. on (b) $t_1 d_m(\eta)$, $t_2 d_m(\eta) \neg(\beta$-conv.) $\mathbf{0}$ so $e_0(t_1 d_m(\beta))(t_2 d_n(\beta))$ β-conv. $\mathbf{0}$. If $\beta \neq \alpha$ w.l.o.g. assume $\delta \in \beta$ and $\delta \notin \alpha$. By the above observation there is an $\eta \in N_m$ such that $d_m(\eta) \in \text{def}^m(\delta)$. By hyp. ind. on (b) $t_1 d_m(\eta)$ β-conv. $\mathbf{0}$ and $t_2 d_m(\eta) \neg(\beta$-conv.) $\mathbf{0}$ so $e_0(t_1 d_m(\eta))(t_2 d_m(\eta))$ β-conv. $\mathbf{1}$. Thus in either case we have (a).

(b). Let $t_3 = \lambda y \cdot r_1(\cdots(\cdot r_{s(n+1)}((\lambda w \mathbf{1})y))\cdots)$. If $\beta \in \gamma$, then for some $t_4 \in \text{def}^n(\beta)$ and some i, $r_i = e_n t_4 y$. By case (a) $e_n t_4 t_1 \beta$-conv. $\mathbf{0}$ so $t_3 t_1$ β-conv. $\mathbf{0}$. If $\beta \notin \gamma$, then for each $r_i = e_n t_4 y$ $t_4 \notin \text{def}^n(\beta)$ so by case (a) $e_n t_4 t_1 \neg (\beta$-conv.)$\mathbf{0}$. Thus in either case we have (b).

The two propositions taken together tell us that our definitions of e_n and $\forall_n$ work correctly. This is summarized in the following

Theorem 1. *Suppose* $A = A(x_1^{n_1}, \ldots, x_m^{n_m})$ *is an* Ω*-formula,* $\alpha_i \in \mathcal{D}_{n_i}$ *and* $t_i \in \text{def}^{n_i}(\alpha_i)$ *for* $1 \leq i \leq m$, *then* $A[\alpha_1, \ldots, \alpha_m]$ *is true* $\Leftrightarrow (\lambda x_1^{\mathbf{N}_n}, \ldots, x_m^{\mathbf{N}_{n_m}} A^*) t_1 \cdots t_m \beta$-conv. $\mathbf{0}$.

Proof. By induction on A.

Basis: A is atomic. This is just the previous proposition case (b)

Induction step. Cases: $A = B \wedge C$, $A = \neg B$. Immediate by hyp. ind.

Case: $A = \forall x^n B$. We have $A[\alpha_1, \ldots, \alpha_m] \Leftrightarrow \forall \beta \in \mathcal{D}_n B[\alpha_1, \ldots, \alpha_m \beta] \Leftrightarrow \forall t \in \text{def}_n (\lambda x_1^{\mathbf{N}_{n_1}} \cdots x_m^{\mathbf{N}_{n_m}} x^{\mathbf{N}_n} B^*) t_1 \cdots t_m t$ β-conv. $\mathbf{0}$ by hyp. ind. $\Leftrightarrow \text{sg} \sum_{\eta \in N_n} (\lambda x_1^{\mathbf{N}_{n_1}} \cdots x_m^{\mathbf{N}_{n_m}} x^{\mathbf{N}_n} B^*) t_1 \cdots t_m d_n(\eta)$ β-conv. $\mathbf{0}$ by the observation

$$(\lambda x_1^{\mathbf{N}_{n_1}} \cdots x^{\mathbf{N}_{n_m}} \text{sg} \sum_{\eta \in N_m} (\lambda x^{\mathbf{N}_n} B^*) d_m(\eta)) t_1 \cdots t_m \beta\text{-conv. } \mathbf{0}$$

$$(\lambda x_1^{\mathbf{N}_{n_1}} \cdots x_m^{\mathbf{N}_{n_m}} A^*) t_1 \cdots t_m \beta\text{-conv. } \mathbf{0}.$$

Corollary. *For each type $\sigma \neq 0 \to 0$ which is the type of a closed term there is a closed term t^σ such that the problem of determining for arbitrary closed terms r of type σ whether $r\beta$-conv. t^σ, $r\beta$-red t^σ, or t^σ is the β-normal form of r cannot be solved in elementary time.* ($0 \to 0$ *is anomalous because it contains only one β-normal closed term, viz. λxx.*)

Proof. The above theorem establishes the corollary for $\sigma = \mathbf{0}$ with $t^\sigma = \mathbf{0}$. Note that for Ω-sentences A, $\neg A$ is true $\Leftrightarrow A^*\beta$-conv. $\mathbf{1}$.

Case: $\sigma = \underbrace{0 \to (\cdots (0 \to 0) \cdots)}_{m}$ for $m > 1$. We have for closed r of type $\mathbf{0}$, $r\beta$-conv. $\mathbf{0} \Leftrightarrow r(\lambda v_0^0 v_1^0) v_2^0 \beta$-conv. $v_2^2 \Leftrightarrow \lambda v_1^0 \cdots v_m r(\lambda v_0^0 v_1^0) v_2^0 \beta$-conv. $\lambda v_1^0 \cdots v_m^0 v_2^0$ so we can set $t^\sigma = \lambda v_1^0 \cdots v_m^0 v_2^0$.

Case: otherwise. We say that σ contains a splinter if there is a closed term t of type σ and a closed term s of type $\sigma \to \sigma$ such that the β-normal forms of $t, st, \ldots, s(\cdots(st)\cdots), \ldots$ are all distinct. It is easy to prove that σ contains a splinter $\Leftrightarrow \sigma$ contains a closed term and σ does not have the form $0 \to (\cdots (0 \to 0) \cdots)$. Suppose σ contains a splinter generated by t and s; we have for closed r of type $\mathbf{0}$, $r\beta$-conv. $\mathbf{0} \Leftrightarrow [\sigma/0] r\beta$-conv. $[\sigma/0]\mathbf{0} \Leftrightarrow ([\sigma/0]r)st\,\beta$-conv. t so we can set $t^\sigma = t$.

6. Extensions and refinements

By a consistent extension Λ^+ of Λ we mean an extension of Λ with a model whose ground domain has ≥ 2 elements (note that Λ^+ need not be closed under the inductive definition of β-conversion and the model need not be extensional). If Λ^+ is an extension of Λ and $\Lambda^+ \vdash \mathbf{0} = \mathbf{1}$, then $\Lambda^+ \vdash v_1^0 = v_2^0$ so Λ^+ is not consistent. Thus if Λ^+ is a consistent extension of Λ, for Ω-sentences A, A is true $\Leftrightarrow \Lambda^+ \vdash A^* = \mathbf{0}$. More generally we have the

Theorem 2. *If σ is the type of a closed term and σ contains no positive occurrence of a subtype of the form $\sigma_1 \to (\sigma_2 \to \sigma_3)$ (see Prawitz [15, p. 43 and read $\to$ for $\supset$]), then there is a closed term t^σ of type σ such that the problem of determining for an arbitrary closed term r of type σ whether r β-conv. t^σ, r β-red t^σ or t^σ is the β-normal form of r cannot be solved in elementary time.*

Our proof of this theorem uses the model theory of Statman [18] and is proved there.

The rank of a type is defined as follows: $\mathrm{rnk}(0) = 0$ and $\mathrm{rnk}(\sigma \to \tau) = \max\{\mathrm{rnk}(\sigma) + 1, \mathrm{rnk}(\tau)\}$. Set $T_n = \{t \in \Lambda : \text{each subterm of } t \text{ has type with rnk} \leq n\}$. It is easy to see (by analysis of the normal form algorithm) that the problem for arbitrary closed terms $t_1, t_2 \in T_n$ of the same type of whether $t_1\beta$-conv. t_2 can be solved in elementary time. By modifying the above construction (using Meyer's result for the monadic predicate calculus instead of Ω; see Meyer [13, p. 478]) it is easy to find an n such that

Proposition 5. *The problem for arbitrary closed $t \in T_n$ of whether t β-conv. $\mathbf{0}$ cannot be solved in polynomial time.*

If F is a finite set of types let $T_F = \{t \in \Lambda$: each subterm of t has type $\in F\}$. By modifying the above construction (using the Meyer–Stockmeyer result for B_ω instead of Ω; see Stockmeyer [19, p. 12]) it is easy to find an F such that

Proposition 6. *The problem for arbitrary closed $t \in T_F$ of whether t β-conv. $\mathbf{0}$ is polynomial-space hard.*

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