3/5: Intro to Optimal Control

Announcements:

- HW2 due today, HW3 out Mon.
- Midterm survey out Mon.

Last Time:

- Zero-order (gradient-free) methods

Today:

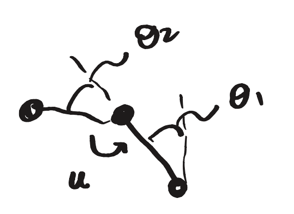
- the optimal control problem
- LQR

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } h(x) = 0, \; g(x) \le 0$$

- Numerical optimization
  - Unconstrained
    - 1st: g.d. $\nabla f(x)$
    - 2nd: Newton $\nabla^2 f(x)$
    - 0th
  - Constrained
    - Convex/QPs
    - penalty
    - interior point

e.g.) (acrobot) $x = [\theta_1, \theta_2, \dot\theta_1, \dot\theta_2]$



Key idea: cost function

$x_{0:T}$, $u_{0:T-1}$ $\quad J(x_{0:T}, u_{0:T-1}) = w_{term} \|x_T\|^2 + \sum_{t=0}^{T-1} w_x \|x_t\|^2 + w_u \|u_t\|^2$

General form:

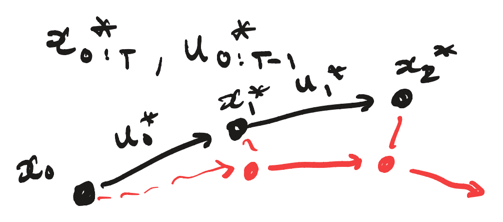
$$\min_{x_{0:T}, u_{0:T-1}} c_T(x_T) + \sum_{t=0}^{T-1} c_t(x_t, u_t)$$

$$\text{s.t. } x_{t+1} = f(x_t, u_t) + \varepsilon_t$$

$$x_0 = x(0)$$

Trajectory optimization

$x^*_{0:T}$, $u^*_{0:T-1}$



"open loop"

Dynamic programming

$u_t = \pi^*(x_t)$

"closed loop"

Linear Quadratic Regulator (LQR)

$$\min_{x_{0:T}, u_{0:T-1}} \tfrac{1}{2} x_T^T Q_T x_T + \sum_{t=0}^{T-1} \tfrac{1}{2} x_t^T Q_t x_t + \tfrac{1}{2} u_t^T R_t u_t$$

$$\text{s.t. } x_{t+1} = A x_t + B u_t \qquad Q_t \succeq 0 \quad R_t \succ 0$$

$$Q_t = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix} \in \mathbb{R}^{n \times n}, \; q_i \ge 0 \qquad R_t = \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_m \end{bmatrix} \in \mathbb{R}^{m \times m}, \; r_j > 0$$

$$x^T Q x = \sum_{i=1}^n q_i x_i^2 \qquad x = [x_1, x_2, \dots x_n]$$

e.g.) $Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 0.1 & \\ & & & 0.1 \end{bmatrix}$ position > velocity

Key insight: LQR is a QP

$$z = [x_0^T \; u_0^T \; x_1^T \; u_1^T \; \dots \; x_T^T]^T \in \mathbb{R}^{Tn + (T-1)m}$$

$$\tfrac{1}{2} z^T P z = \tfrac{1}{2} z^T \begin{bmatrix} Q_1 & & & 0 \\ & R_1 & & \\ & & \ddots & \\ 0 & & & Q_T \end{bmatrix} z$$

$x_{t+1} = A x_t + B u_t \qquad (A x_t + B u_t - x_{t+1} = 0)$

$$[\dots \; A \;\; B \;\; -I \; \dots] z = 0$$

$$\begin{bmatrix} I & 0 & \cdots & & 0 \\ A & B & -I & & \\ & & A & B & -I \\ 0 & & & \ddots & \\ & & & A \; B & -I \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ x_T \end{bmatrix} = \begin{bmatrix} x(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C \qquad z \qquad = d$$

$$\min_z \tfrac{1}{2} z^T P z = f(x) \qquad \nabla^2 f(x) \succeq 0 \checkmark$$

$$\text{s.t. } Cz = d \quad = h(x) = 0 \checkmark$$

Special structure: - sparse
- equality-constrained QP

Lagrangian: $\mathcal{L}(z, \lambda) = \tfrac{1}{2} z^T P z + \lambda^T (Cz - d)$

z^*, λ^* Stationarity $\nabla_z \mathcal{L}(z^*, \lambda^*) = 0$

Feasibility $\nabla_\lambda \mathcal{L}(z^*, \lambda^*) = 0$

$$\nabla_z \mathcal{L} = P z^* + C^T \lambda^* = 0$$

$$\nabla_\lambda \mathcal{L} = C z^* - d = 0$$

$$\begin{bmatrix} P & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} z^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} 0 \\ +d \end{bmatrix}$$

"KKT system"
linear solve
$O([T(n+m)]^3)$

$T = 2$

$$\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ \\ \\ \\ \end{array} \left[\begin{array}{ccccc|ccc} Q_0 & & & & & I & A^T & \\ & R_0 & & & & & B^T & \\ & & Q_1 & & & & -I & A^T \\ & & & R_1 & & & & B^T \\ & & & & Q_2 & & & -I \\ \hline I & & & & & & & \\ A & B & -I & & & & 0 & \\ & & A & B & -I & & & \end{array}\right] \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ x_2 \\ \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ x(0) \\ 0 \\ 0 \end{bmatrix}$$

(1) $Q_1 x_1 - \lambda_1 + A^T \lambda_2 = 0$

(2) $R_1 u_1 + B^T \lambda_2 = 0$

(3) $Q_2 x_2 - \lambda_2 = 0 \Rightarrow \lambda_2 = Q_2 x_2$

(2) $R_1 u_1 + B^T Q_2 x_2 = 0$

dyn: $R_1 u_1 + B^T Q_2 (A x_1 + B u_1) = 0$

$$u_1 = -(R_1 + B^T Q_2 B)^{-1} B^T Q_2 A x_1$$

$$= -K_1 x_1$$

(1) $Q_1 x_1 - \lambda_1 + A^T \lambda_2 = Q_1 x_1 - \lambda_1 + A^T Q_2 x_2 = 0$

dyn: $Q_1 x_1 - \lambda_1 + A^T Q_2 (A - BK_1) x_1 = 0$

$$\lambda_1 = (Q_1 + A^T Q_2 (A - BK_1)) x_1$$

$$P_1 x_1$$

$u_t = -K_t x_t, \quad \lambda_t = P_t x_t$

"Riccati recursion"
$O(T)$

$$P_T = Q_T, \quad K_t = (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

$$P_t = Q_t + A^T P_{t+1} (A - B K_t)$$