3/24: State estimation

Announcements:
- HW3 due Thurs (3/26)
- HW4 postponed until → SB
- project feedback EOW

Last time:
- sampling MPC

Today:
- state estimation

Recall:

$$\min_{x_{0:T}, u_{0:T}} \; c_T(x_T) + \sum_t c_t(x_t, u_t) \qquad x_0 \text{ init. state}$$

s.t. $x_{t+1} = f(x_t, u_t)$

$x_0 = x(0)$

$x_t \in \mathcal{X}_t = \mathbb{R}^n \setminus \mathcal{O}$

$u_t \in \mathcal{U}_t$

e.g.) (drone)

$x_t = [\, p_t, R_t, v_t, \omega_t \,]$

$p_t \in \mathbb{R}^3 \quad R_t \in SO(3) \quad v_t \in \mathbb{R}^3 \quad \omega_t \in \mathbb{R}^3$

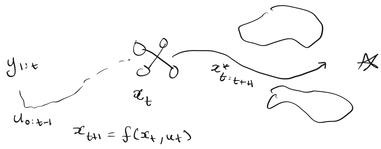
$y_t = [\, \text{IMU}, \text{GPS}, \text{camera (stereo, optical flow)} \,]$ (lidar)

$p_t + \varepsilon_t \qquad I_t \in \mathbb{R}^{4 \times w \times 3}$

$w_t + \varepsilon_t$

based

$\dot{v}_0 = g(u_t, R)$ acceleration



$x_{t+1} = f(x_t, u_t)$

models of our system:

$x_{t+1} \sim p(\cdot \mid x_t, u_t)$

$y_{t+1} \sim p(\cdot \mid x_{t+1}, u_{t+1}) = p(\cdot \mid x_{t+1})$

Implicitly, we assume conditional ind.

meas. independent of previous timesteps

$$p(y_{t+1} \mid x_{0:t+1}, u_{0:t}, y_{0:t}) = p(y_{t+1} \mid x_{t+1})$$

"markov" $\quad p(x_{t+1} \mid x_{0:t}, u_{0:t}, y_{0:t}) = p(x_{t+1} \mid x_t, u_t)$

e.g.) plane w/ fuel

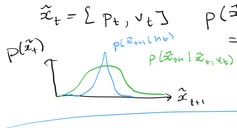
$x_t = [\, p_t, v_t, f_t \,]$

$p_{t+1} = p_t + \Delta t\, v_t + w^p_t \qquad w^p_t \sim \mathcal{N}(0, Q^p)$

$v_{t+1} = v_t + \frac{\Delta t}{m f_t} u_t + w^v_t$

$f_{t+1} = f_t - \alpha \Delta t$

$\tilde{x}_t = [\, p_t, v_t \,] \qquad p(\tilde{x}_{t+1} \mid \tilde{x}_t, u_t) = p(\tilde{x}_{t+1} \mid \tilde{x}_{0:t}, u_{0:t})$



Recall: probability basics

i) For a random variable X, we can describe its pdf

$p(X), \quad p(x) \geq 0, \quad \int_x p(x)\,dx = 1$

ii) Independence

For two RVs X, Y, we say X is independent of Y if $p(x, y) = p(x)p(y)$

iii) Conditional dist.

$$p(x = X \mid y = Y) = \frac{p(x, y)}{p(y)}$$

iv) Bayes' rule

$$p(x \mid y) = \frac{p(y \mid x)\, p(x)}{p(y)}$$

Typically, we approach state estimation using Bayesian estimation

Inputs: $u_{0:t-1}$

Prior: $p(x_0)$

Measurements $y_{1:t}$

$p(x_t \mid u_{0:t-1}, y_{1:t}) = b(x_t)$ "belief"

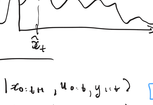
$$p(x_{t+1} \mid u_{0:t}, y_{1:t+1}) \propto p(y_{t+1} \mid x_{t+1}) \int_{x_t} p(x_{t+1} \mid x_t, u_t)\, b(x_t)\, dx_t$$

$b(x_0) = p_0 \longrightarrow \dots \longrightarrow b(x_t)$

Maximum a posteriori (MAP) inference

$$\hat{x}_t = \arg\max_{x_t} \; p(x_t \mid u_{0:t-1}, y_{1:t})$$

$$\boxed{\hat{x}_{0:t} = \arg\max_{x_{0:t}} \; p(x_{0:t} \mid u_{0:t}, y_{1:t})}$$



$p(x_{0:t+1} \mid u_{0:t}, y_{1:t+1}) \propto p(y_{t+1} \mid x_{0:t+1}, u_{0:t}, y_{1:t}) \cdot p(x_{0:t+1} \mid u_{0:t}, y_{1:t})$ [Bayes' rule]

$p(y_{t+1} \mid x_{0:t+1}, u_{0:t}, y_{1:t}) = p(y_{t+1} \mid x_{t+1})$ [independence y_{t+1}]

$p(x_{0:t+1} \mid u_{0:t}, y_{1:t}) = p(x_{t+1} \mid x_{0:t}, u_{0:t}, y_{1:t}) \cdot p(x_{0:t} \mid u_{0:t}, y_{1:t})$ [$p(a|b) = \frac{p(a,b)}{p(b)}$]

$= p(x_{t+1} \mid x_t, u_t) \cdot p(x_{0:t} \mid u_{0:t-1}, y_{1:t})$ [independence x_{t+1}]

$$p(x_{0:t} \mid u_{0:t-1}, y_{1:t}) = p(x_0) \prod_{t=0}^{t-1} p(y_{t+1} \mid x_{t+1}) \, p(x_{t+1} \mid x_t, u_t)$$ [telescoping]

$$\hat{x}_{0:t} = \arg\min_{x_{0:t}} \; -\log p(x_{0:t} \mid u_{0:t-1}, y_{1:t})$$

$$= \arg\min_{x_{0:t}} \left\{ -\log p(x_0) + \sum_{t=0}^{t-1} -\log p(x_{t+1} \mid x_t, u_t) - \log p(y_{t+1} \mid x_{t+1}) \right.$$