1/29: SO(3), rotations

Announcements:

- intro survey out, due today
- HW1 out today, due 2/12

Last time:

- linear systems
- autodiff

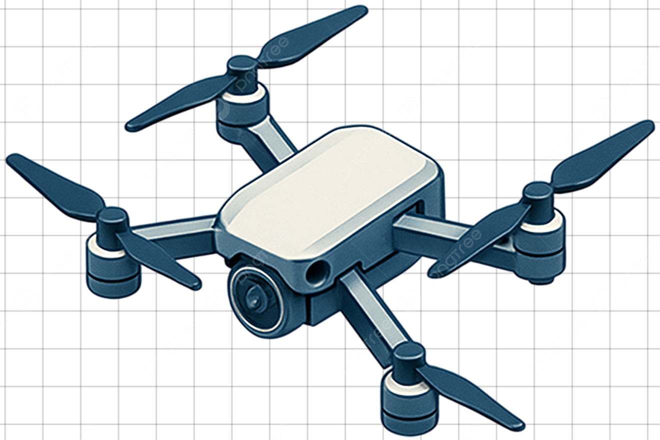
Today:

- defining 3D rotations
- tangent space, exp / log map
- "addition"/ "subtraction" on SO(3)

$$\dot{x} = f(x, u) \Rightarrow x_{k+1} = \bar{f}(x_k, u_k$$

$$\boxed{\dot{x} = Ax + Bu} \approx \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial u}\delta u$$

stability :)



$x = [p, R, v, w]$ state

$u = [f_1, f_2, f_3, f_4]$ input

$$R_{k+1} = R_k \oplus \Delta t \, \delta R_k$$

1. why are rotations special

2. define $\oplus$ $\ominus$

Def 1 (SO(3)) "special orthogonal"

A matrix $R \in \mathbb{R}^{3\times 3}$ is in $SO(3)$ if:

☆ i) $R^T R = I$

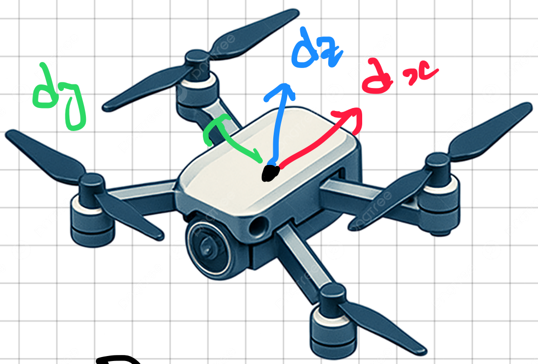
ii) $\det(R) = +1$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

e.g. 1

to ↘ ↙ from

$${}^{w}R^{D} \in SO(3) = \begin{bmatrix} \hat{d}_x^{w} & \hat{d}_y^{w} & \hat{d}_z^{w} \end{bmatrix}$$ ☆

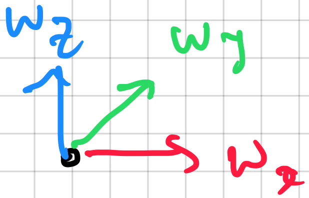


D

i) orthonormal $\|d_i\| = 1$

$d_i^T d_j = 0 \quad i \neq j$

w



$$({}^{w}R^{D})^T ({}^{w}R^{D}) = I$$

Two interpretations of ${}^{w}R^{D}$

i) $${}^{w}R^{D} = \begin{bmatrix} d_x^{w} & d_y^{w} & d_z^{w} \end{bmatrix}$$

"representing orientation"

ii) operator $D \Rightarrow w$

$$\boxed{v^{w} = {}^{w}R^{D} \, v^{D}}$$

$R(t)$

Consider $R \in SO(3)$

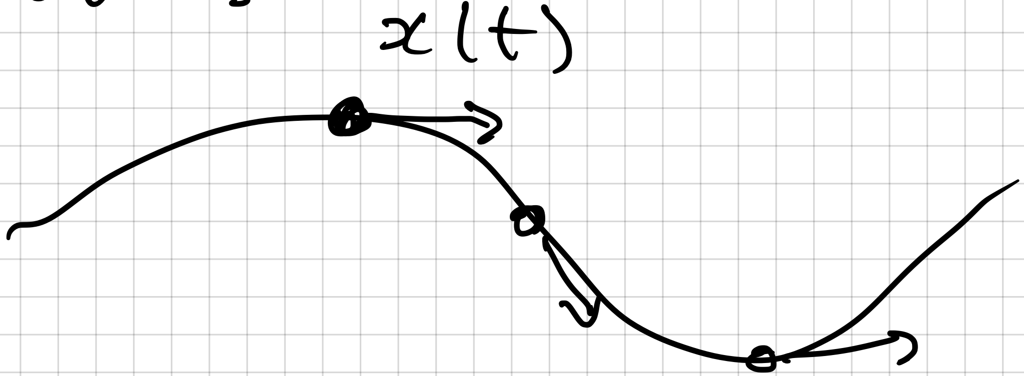
$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad \begin{aligned} R^T R &= I \\ \det(R) &= +1 \end{aligned}$$

$$DOF = 9 - \#\text{constraints} = 9 - 6 = 3$$

$$R = [r_x \ \ r_y \ \ r_z]$$

$$\left.\begin{aligned} r_x^T r_y &= 0 \qquad & r_x^T r_x &= 1 \\ r_y^T r_z &= 0 \qquad & r_y^T r_y &= 1 \\ r_z^T r_x &= 0 \qquad & r_z^T r_z &= 1 \end{aligned}\right\} \text{ 6 constraints}$$

Rotation kinematics



$R(t)$

$$R(t)^T R(t) = I$$

$$\frac{d}{dt}\left(R^T R = I\right)$$

$$\dot{R}(t)^T R(t) + R(t)^T \dot{R}(t) = 0$$

$$\underline{\dot{R}^T R} = -\left(R^T \dot{R}\right) = -\left(\underline{\dot{R}^T R}\right)^T$$

$$A = \dot{R}^T R$$

$$A^T = -A$$

"skew-symmetric"

$$R^T \dot{R} = -(R^T \dot{R})^T$$ "skew-symmetric"

$$v^\wedge$$

$$\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \cong \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \theta \hat{u}$$

$$\{ A \in \mathbb{R}^{3\times 3} \mid A^T = -A \} \cong \mathbb{R}^3$$

$R\cdot$ $$R^T \dot{R} = {}^{R\cdot} v^\wedge$$

$$R^T R = R R^T = I$$

$$\boxed{\dot{R} = R v^\wedge}$$

$$\dot{R}$$

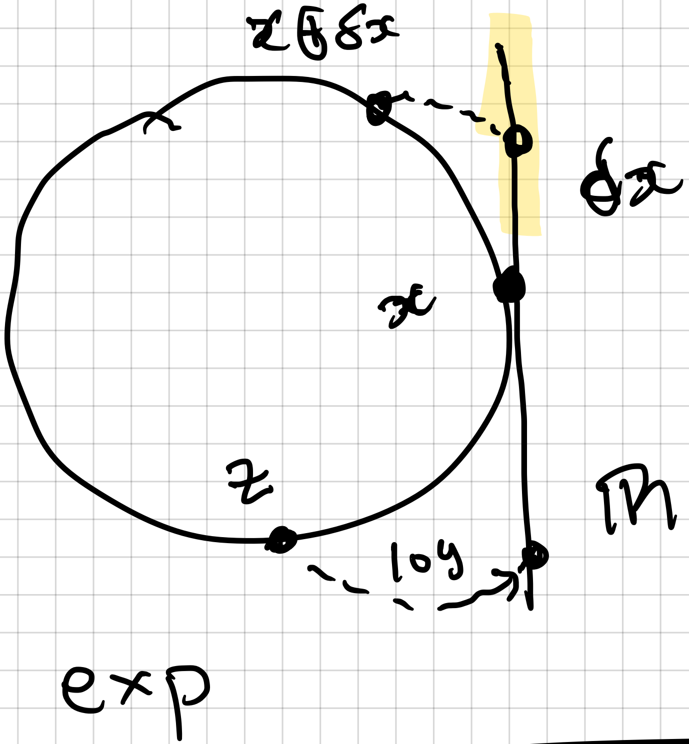
Upshot: $so(3)$ "behaves locally" like $\mathbb{R}^3$

Goal: Find ways to apply perterbation

$$R_{k+1} = R_k + \Delta t \, \cancel{\delta R} \qquad \delta\omega \in \mathbb{R}^3$$

$\uparrow$ 9

$$= R_k \oplus \Delta t \, \delta\omega \qquad \text{"exp" / "log"}$$



$x \in \mathbb{R}^2$ $\quad \{x \mid x^T x = 1\}$

$\delta x = \theta$

$x + \delta x \notin C$

"bad picture"

"line" $\quad \underline{so}(3) = \{\omega^\wedge \mid \omega \in \mathbb{R}^3\}$ $\quad \wedge$: "promotion"

$\exp : \underline{so}(3) \to SO(3)$ $\quad$ matrix exponential

expm

$\log : SO(3) \to \underline{so}(3)$

Prop 1 (Rodrigues)

$$\exp(\underline{w}^\wedge) = \underline{I} + u^\wedge \sin\theta + (u^\wedge)^2 (1-\cos\theta)$$

$$\theta = \|w\| \qquad u = \frac{w}{\|w\|}$$

Def 1 (Box plus)

For $\underline{w \in \mathbb{R}^3}$, $\quad R \oplus w = R \cdot \exp(w^\wedge)$

Def 1 (Box minus)

For $R_1, R_2 \in SO(3)$ $\quad R_1 \ominus R_2 = \log(R_1^T R_2)^\vee$