1/22: Dynamics & integrators

Announcements:

- intro survey out, due Thurs (1/29)
- HW0 out, not graded
- HW1 out next Thurs (1/29)
- Autograder issue fixed :)

Today:

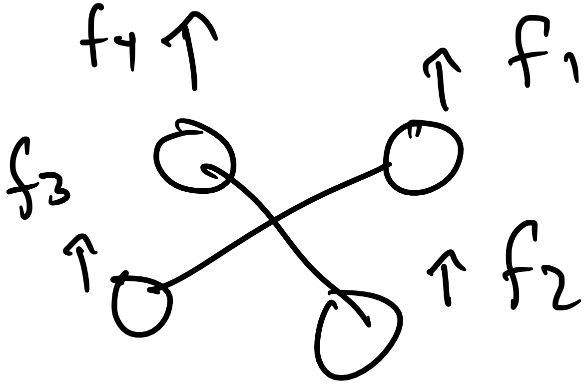
- what is a dynamical system?
- numerical integrators
- demo in code

OH:

Preston: CIS 459
Thurs 2-4p

Samuel: Malott 304H
Thurs 5-7p

Q: What is a dynamical system?



drone

x: state - description of where the drone is + how it's moving

u: inputs we provide to the system

$x = [p \in \mathbb{R}^3, \dot{p} \in \mathbb{R}^3$
$R \in SO(3), \omega \in \mathbb{R}^3]$

$\dot{x} = f(x, u)$
$\equiv \frac{dx}{dt}$

$u = [f_1, f_2, f_3, f_4] \in \mathbb{R}^4$

why is continuous time hard?

$$\min_{x(t)} \int_0^T x(\tau)\, d\tau$$

$x(t)$, $x(t+\varepsilon)$ ...

"indirect methods"

$\Downarrow$

calculus of variables

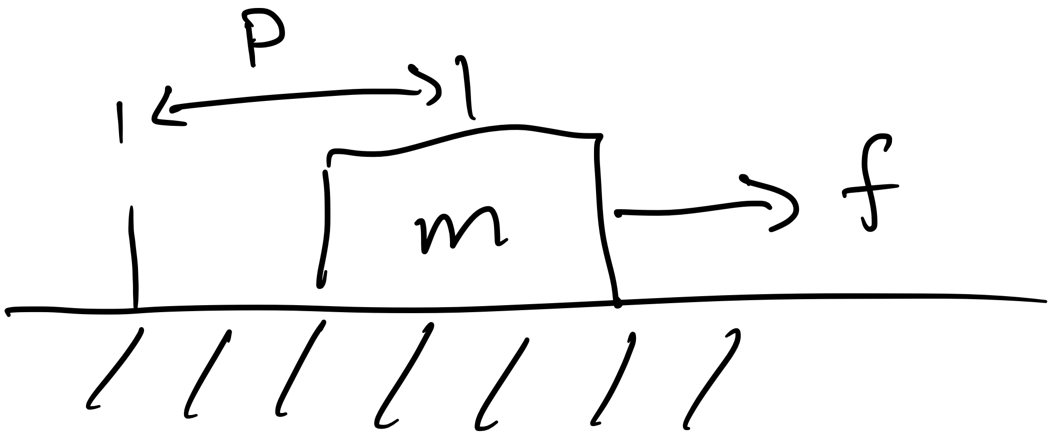
$x(t)$

"direct methods"

$\Downarrow$

$$x_{k+1} = f(x_k, u_k)$$

e.g. 1 (Double integrator)



$$\boxed{\dot{x} = Ax + Bu}$$

$$x = \begin{bmatrix} p & \dot{p} \end{bmatrix}$$

$$u = \begin{bmatrix} f \end{bmatrix}$$

$$f = ma = m\ddot{p}$$

$$\overset{?}{\Longrightarrow} \quad \dot{x} = f(x, u)$$

$$\dot{x} = \dot{\begin{bmatrix} p \\ \dot{p} \end{bmatrix}} = \begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{p} \\ \frac{1}{m} \cdot f \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

e.g. | (Humanoid robot)

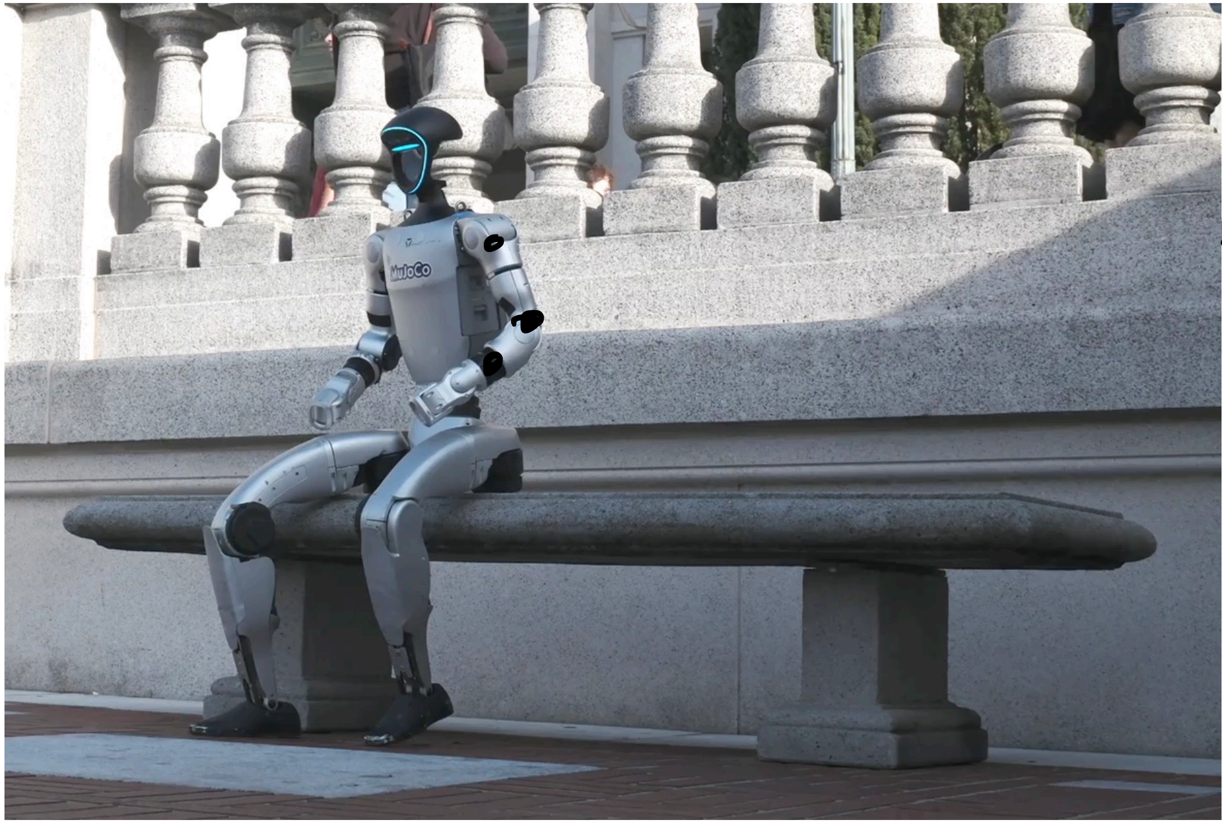
"manipulator eqs."

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = B\tau + F_{ext}}$$

τ : inputs

contact

$J(q)^T \lambda$



$$x = [\theta, \dot{\theta}, p, \dot{p}, R, \omega]$$

$$q = [\theta, p, R]$$

$$\dot{q} = [\dot{\theta}, \dot{p}, \omega]$$

2) Numerical integration

$$\dot{x} = f(x, u) \implies x_{k+1} = \tilde{f}(x_k, u_k)$$

$$x_{k+1} = x(t + \Delta t)$$

$$x_k = x(t)$$

Fundamental theorem:

$$x(t+\Delta t) = x(t) + \int_t^{t+\Delta t} f(x(\tau), u(\tau))\, d\tau$$

<u>Assumption:</u> $u(\tau) = u \qquad \tau \in [t, t+\Delta t]$

"zero-order hold"

Simplest method: Euler's method

$$x(t+\Delta t) = x(t) + \Delta t \cdot f(x(t), u(t))$$

Pros: simple! function calls = 1

Cons: local accuracy $O(\Delta t^2)$

high error

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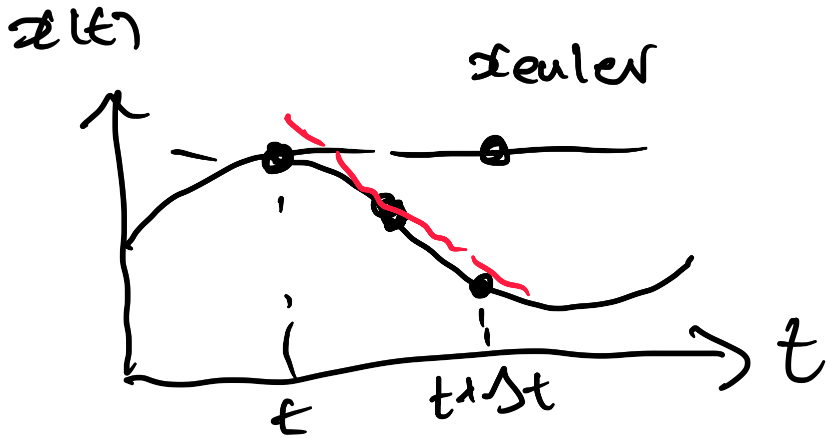
midpoint method

\# evals: 2

error: $O(\Delta t^3)$

$$x(t+\Delta t) = x(t) + \Delta t \cdot \underline{f\left(x\left(t+\frac{\Delta t}{2}\right), u\right)}$$



$$K_1 = x(t) + \frac{\Delta t}{2} f(x(t), u)$$

$$x(t+\Delta t) \approx x(t) + \Delta t \cdot f(K_1, u)$$

In general, Runge-Kutta RK(4)

$$x(t+\Delta t) \approx x + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

evals: 4

error: $O(\Delta t^5)$

$$k_1 = f(x(t), u)$$

$$k_2 = f(x + \frac{\Delta t}{2} k_1, u)$$

$$k_3 = f(x + \frac{\Delta t}{2} k_2, u)$$

$$k_4 = f(x + \Delta t\, k_3, u)$$