2/3: $SO(3)$ (cont'd), unconstrained opt.

Announcements:

- HW1 out, due 2/12 (Thurs)

Last time:

- Defining $SO(3)$
- exp / log maps

Today:

- Gradients, integration on $SO(3)$
- Fundamentals of unconstrained opt.

Def | 3D Rotation matrix $SO(3)$

A matrix $R \in \mathbb{R}^{3\times 3}$ is a rotation matrix if:

$$R = [\hat{x}_b^w \; \hat{y}_b^w \; \hat{z}_b^w]$$

i) $R^T R = I$

ii) $\det(R) = +1$

$R(t)$: trajectory

$$w^\wedge = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

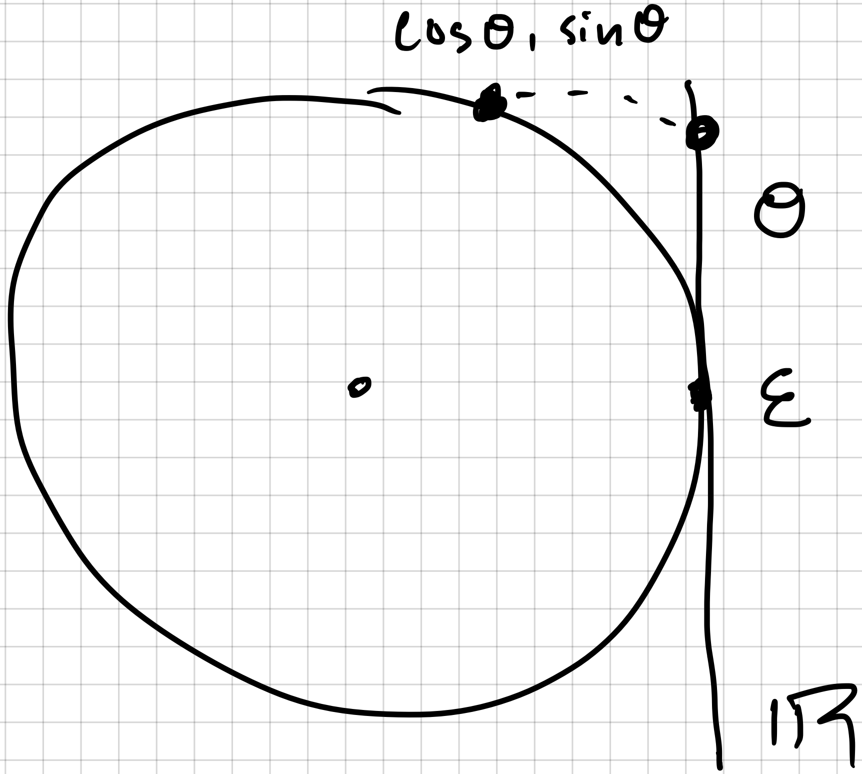
$$\dot{R}(t) = R(t)\, w^\wedge$$

$$(w^\wedge)^\vee \equiv w$$

$$w \times r = w^\wedge \cdot r$$

$$w, r \in \mathbb{R}^3$$

Last time, we drew a picture



$$S^1 = \{ x \in \mathbb{R}^2 \mid x^T x = 1 \}$$

$$\frac{d}{dt}(x^T x) \Rightarrow 2\dot{x}^T x = 0$$

$$T_x = \{ v \mid v^T x = 0 \}$$

"looks locally linear"

$$\varepsilon = [1, 0]^T$$

$$\exp : T_\varepsilon \longrightarrow S^1 = \exp(\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\log : S^1 \longrightarrow T_\varepsilon = \exp^{-1} = \log\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \mathrm{atan2}(x, y)$$

For $SO(3)$, we need the same game

$\mathcal{E} = I_3$

$\exp : \underline{so}(3) \to SO(3)$: matrix exponential
expm

$\log : SO(3) \to \underline{so}(3)$: matrix log

<u>Def</u> $\underline{so}(3) = \{ S \in \mathbb{R}^{3\times 3} \mid S + S^T = 0 \}$

<u>Def</u> (Box plus)

$\oplus : SO(3) \times \mathbb{R}^3 \longrightarrow SO(3)$

$R \oplus \delta R = R \cdot \exp(\delta R^{\wedge})$

Def 1 (Box minus)

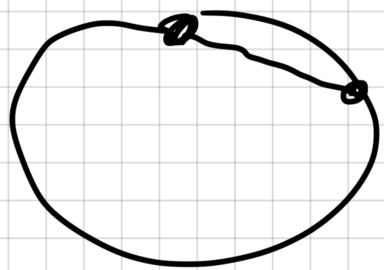
$$R(t) = R_1 \cdot \exp(t \cdot \delta R_{12})$$

$$R(0) = R_1, \quad R(1) = R_2$$

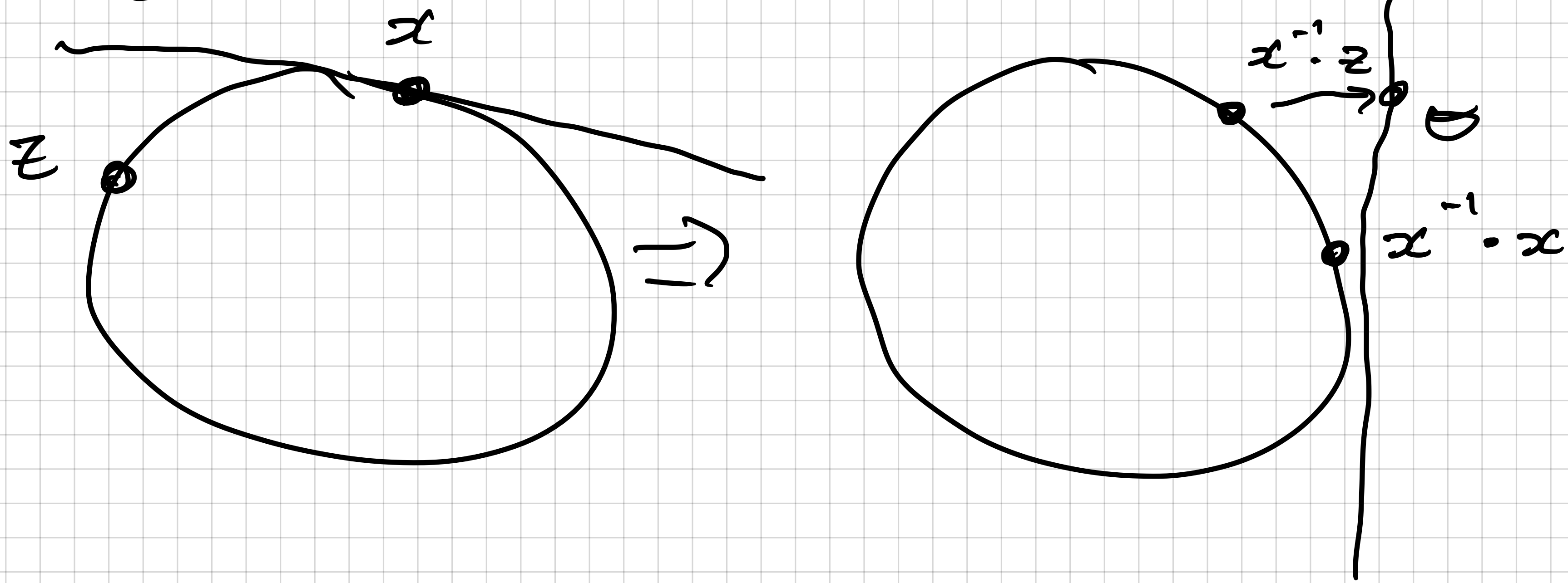
$$\ominus : SO(3) \times SO(3) \longrightarrow \mathbb{R}^3$$

$R_1, R_2 \in SO(3)$, $\quad R_1 \ominus R_2 = \log(R_1^T R_2)^\vee$

Bad example: $\| R_1 - R_2 \|_F$



$x \ominus z$



e.g.) (Derivatives on $SO(3)$)

Recall: $f : \mathbb{R}^n \to \mathbb{R}^m$

$$\frac{\partial f}{\partial x} = \mathbb{R}^{m \times n}, \quad [j_1, \ldots, j_n]$$

$$j_i = \lim_{h \to 0} \frac{f(x + h e_i) - f(x)}{h}$$

For: $f : SO(3) \to \mathbb{R}^m$,

can define Jacobian $\frac{\partial f}{\partial R} = \mathbb{R}^{m \times 3} = [j_1, j_2, j_3]$

$$j_i = \lim_{h \to 0} \frac{f(R \oplus h e_i) - f(R)}{h}$$

$$j_i = \lim_{h \to 0} \frac{f(R \oplus h e_i) - f(R)}{h}$$

$$= \lim_{h \to 0} \frac{f(R \cdot \exp(h e_i)) - f(R)}{h}$$

$$= \left. \frac{\partial \tilde{f}}{\partial x} \right|_{x=0} \qquad \tilde{f} = f(R \cdot \exp(x))$$

TPS: An easier way to optimize rotations would be to parametrize them with exp

$$\theta \in \mathbb{R}^3 \Rightarrow R = \exp(\theta)$$

$$\min_{R \in SO(3)} f(R) \Rightarrow \min_{\theta \in \mathbb{R}^3} f(\exp(\theta))$$

Pros: Simple, remain on $SO(3)$

Cons: $\nabla_\theta \exp(\theta)$ poorly conditioned as $\|\theta\| \rightarrow \pi$.