2/26: Convexity & SQP

Announcements:
- HW 2 due next Thurs (3/5)
- Project feedback EOW

Last time:
- KKT interp, Lagrangians
- Penalty methods, Aug. Lagrangians

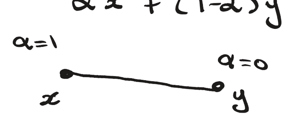
Today:
- Convex functions/sets
- Sequential Quadratic Programming (SQP)

Key idea: Convex combination

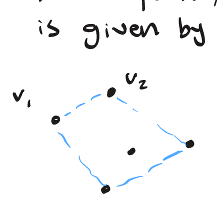
Def (convex comb)

For two points $x, y \in \mathbb{R}^n$, a convex combination is given by

$$\alpha x + (1-\alpha) y, \quad \text{for } \alpha \in [0,1]$$



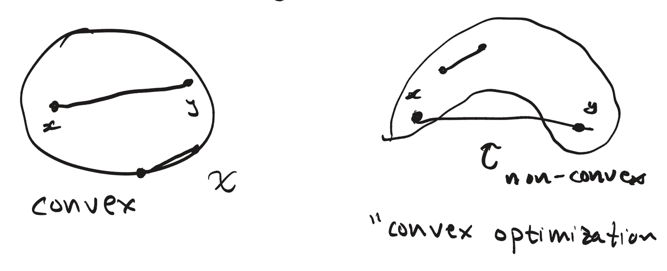
For N points, a convex combination is given by $\sum \alpha_i v_i$, $\alpha_i \geq 0$, $\sum_{i=1}^{N} \alpha_i = 1$



Def (Convex Set)

A set $\mathcal{X} \subset \mathbb{R}^n$ is convex if for any $x, y \in \mathcal{X}$

$$\alpha x + (1-\alpha) y \in \mathcal{X}, \quad \text{for } \alpha \in [0,1]$$



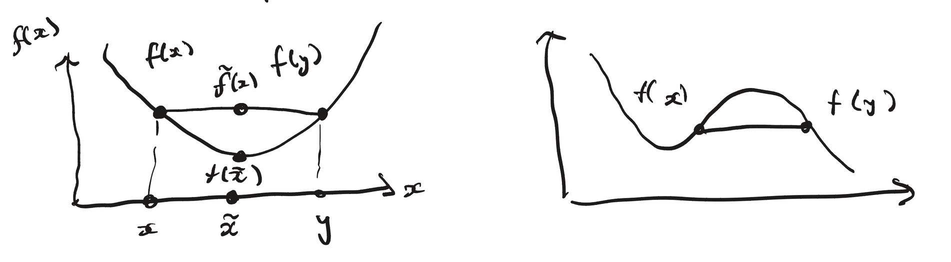
"convex optimization" – Boyd

Def (Convex function)

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if for $x, y \in \text{dom} f$,

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$

"bowl-shaped"



Def (Convex optimization)

An optimization
$$\min_x f(x) \quad \text{s.t.} \quad h_i(x) = 0, \quad g_j(x) \leq 0$$

is convex if:

i) $f(x)$ is convex

ii) $h_i(x)$ is affine $\quad (h_i(x) = a_i^T x - b_i = 0)$, $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}^n$

iii) $g_j(x)$ is convex

Aside: (ii) & (iii) imply the feasible set is convex.

Why do we care?

1. If our problem is (smooth) and convex, a point x^*, λ^*, μ^* satisfying KKT is globally optimal.
2. There exist very efficient solvers for most classes of convex optimizations.

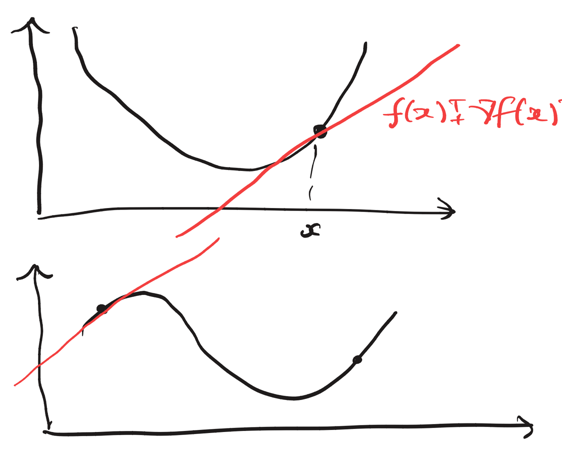
Software: CVX (modeling)

For "reasonably sized" problems → miliseconds

Prop (First-order conditions)

For a differentiable function $f: \mathbb{R}^n \to \mathbb{R}$, f is convex iff $\text{dom} f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x), \quad \forall x, y \in \text{dom} f$$

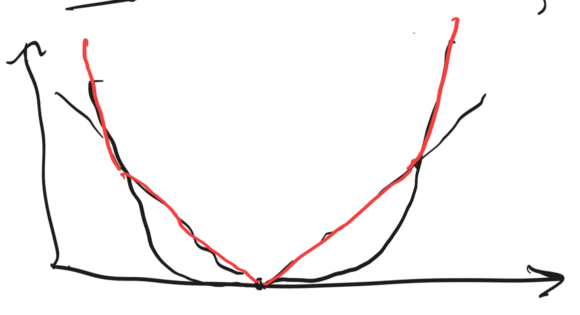


Prop (Second-order convexity conditions)

A twice-differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ is convex iff $\nabla^2 f(x) \succeq 0 \quad \forall x \in \text{dom} f$.

(positive curvature everywhere)

e.g. (non-smooth, convex)



$$\max \{ |x|, x^2 \}$$

e.g. (exp)

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x \geq 0 \checkmark$$

Examples of convex optimizations

i) Linear Program (LP)

$$\min_{x \in \mathbb{R}^n} c^T x \quad \text{s.t.} \quad Ax = b, \quad Gx \leq h$$

CPLEX, Gurobi

operations, economics

ii) Quadratic Programs (QPs)

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} x^T Q x + q^T x \quad \text{s.t.} \quad Ax = b, \quad Gx \leq h$$

convex if $Q \succeq 0$

OSQP

Differential Ite,

iii) Semidefinite Programs (SDP)

$$\min_{S \in \mathbb{R}^{n \times n}} f(S) \quad \text{s.t.} \quad S \succeq 0$$

Sequential Quadratic Programming (SQP)

Recall: Newton's method

$$\min_{d \in \mathbb{R}^n} f(x_k + d) \approx f(x_k) + \nabla f(x_k)^T d + \frac{1}{2} d^T \nabla^2 f(x_k) d$$

SQP:
$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \leq 0$$

(f(x) ← quad; constraints ← linear)

$x_k \Rightarrow d$? maximally decreases f, s.t. our constraints

$$\min_{d \in \mathbb{R}^n} \frac{1}{2} d^T \nabla^2 f(x_k) d + \nabla f(x_k)^T d$$
$$\text{s.t.} \quad h(x_k) + \nabla h(x_k)^T d = 0$$
$$g(x_k) + \nabla g(x_k)^T d \leq 0$$

($+\lambda I$ above $\nabla^2 f(x_k)$)

1. convexity?
2. feasibility?

SNOPT

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1. ∃ a "nice" class of nonlinear programs that have efficient solvers to global opt.
   ↳ convex optimizations
2. we can use convex optimization to find "inner step" for QP

$x_0 \to x_1 \to \cdots \to x^*$ SNOPT