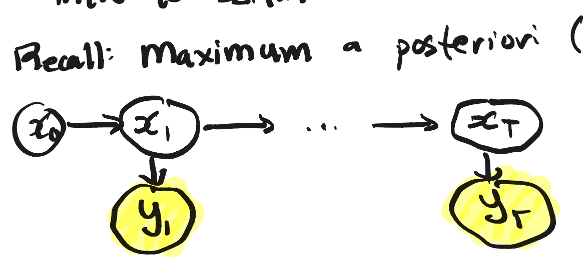
4/7: Factor graph basics

Announcements:
- HW4 out today, due 4/23
- project workshop, 4/28

Last time:
- state est. via Gauss-Newton

Today:
- factor graphs
- intro to SLAM

Recall: maximum a posteriori (MAP) inference



states: $x_{0:T}$
inputs: $u_{0:T-1}$
measurements: $y_{1:T}$

$$\max_{x_{0:T}} p(x_{0:T} \mid u_{0:T-1}, y_{1:T})$$

Given:
- prior: $p(x_0)$
- dynamics model: $p(x_{t+1} \mid x_t, u_t)$
- measurement model: $p(y_{t+1} \mid x_{t+1})$

Main result: when dynamics/meas. Gaussian,

MAP problem becomes:

$$\min_{x_{0:T}} -\log p(x_0) + \sum_{t=0}^{T-1} -\log p(x_{t+1} \mid x_t, u_t) - \log p(y_{t+1} \mid x_{t+1})$$

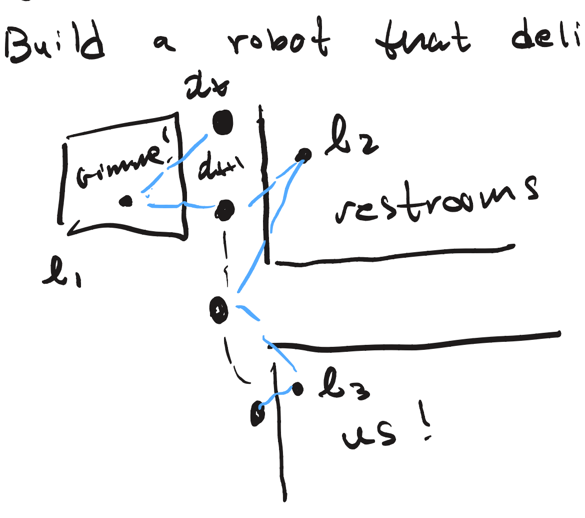
↳

$$\min_{x_{0:T}} \sum_{i=1}^{N} \frac{1}{2} r_i(x_{0:T})^T r_i(x_{0:T})$$

nonlinear least-squares

e.g.) Simultaneous Localization and Mapping (SLAM)

Build a robot that delivers coffee from Gimme!



SLAM: jointly estimate robot pose $x_t \in SE(3)$ and landmarks $p_i \in \mathbb{R}^3$

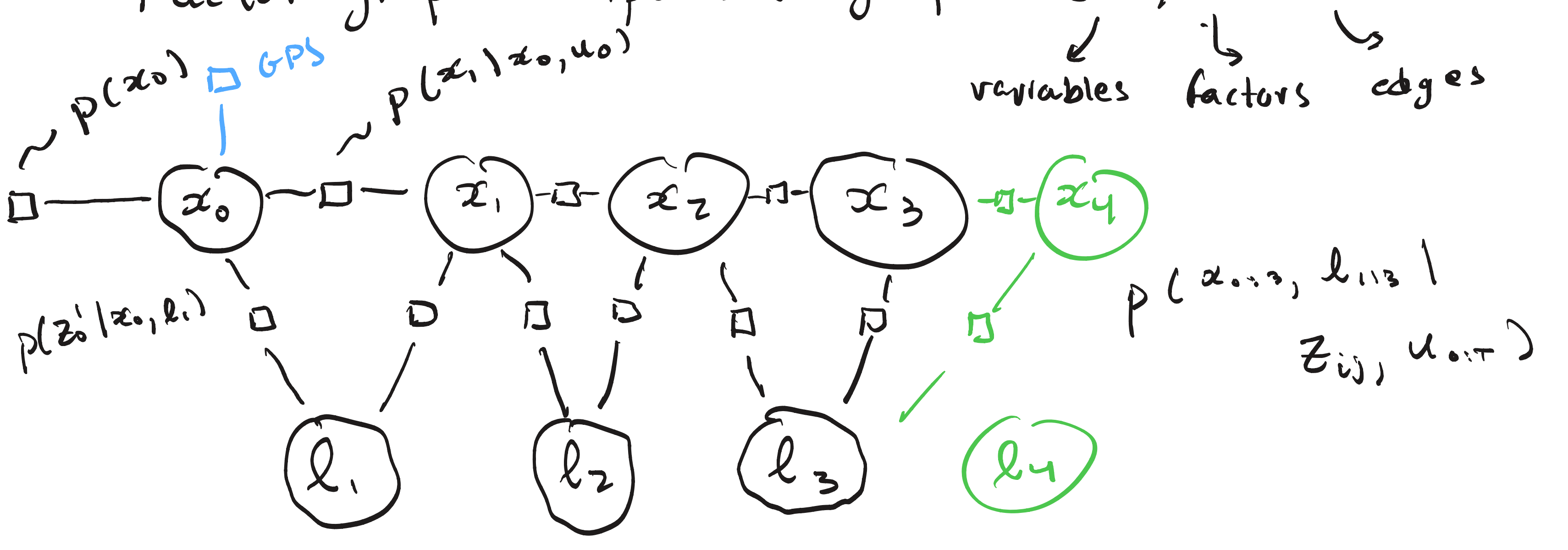
Typically: landmarks from camera / lidar "front end"

measurements: landmark j @ time t,

$$z_t^j = T_t^{-1} \cdot p_j \quad \text{(body-frame location)}$$

Traditionally: $X = [x_{0:T}, p_i^t]$

Factor graph: bipartite graph (V, F, E) — V: variables, F: factors, E: edges



$$\min_{V} \sum_{f \in F} l_f(V)$$

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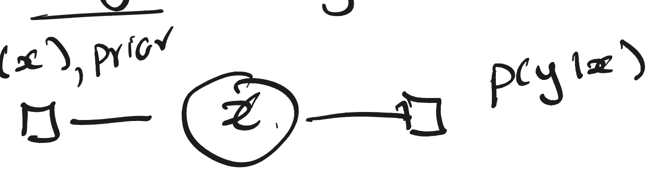
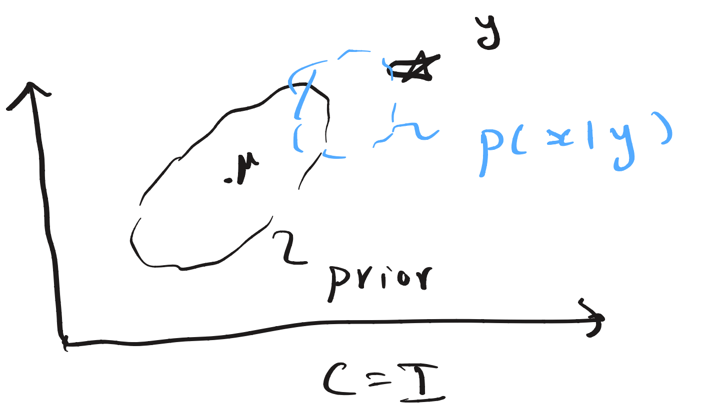
e.g.) (toy factor graph)



$$p(x) = N(\mu, \Sigma)$$
$$p(y \mid x) = N(Cx, R)$$
$$↳ \; y = Cx + v, \quad v \sim N(0, R)$$



$$\min_{x} -\log p(x) - \log p(y \mid x)$$

$$↳ \min_{x} \frac{1}{2} r_p(x)^T r_p(x) + \frac{1}{2} r_m(x)^T r_m(x)$$

$$r_p = \Sigma^{-1/2}(x - \mu)$$
$$r_m = R^{-1/2} C x - R^{-1/2} y$$

Recall: normal equations, given guess $x^0 = \mu$

$$(J_p^T J_p + J_m^T J_m)\delta x = -J_p^T r_p - J_m^T r_m$$

$$J_p = \frac{\partial r_p}{\partial x} = \Sigma^{-1/2} \qquad J_m = R^{-1/2} C$$

$$(\Sigma^{-1} + C^T R^{-1} C)\delta x = 0 - C^T R^{-1}(C\mu - y)$$

$$\delta x = (\Sigma^{-1} + C^T R^{-1} C)^{-1} C^T R^{-1}(y - C\mu)$$

Prop! (Woodbury)

For $M \in \mathbb{R}^{m \times m}$, $N \in \mathbb{R}^{n \times n}$ invertible, $U \in \mathbb{R}^{m \times n}$, $V \in \mathbb{R}^{n \times m}$

$$(M + UNV)^{-1} = M^{-1} - M^{-1} U (N^{-1} + V M^{-1} U)^{-1} V M^{-1}$$

Apply: $M = \Sigma^{-1}$, $N = R^{-1}$, $U = C^T$, $V = C$

$$\delta x = (\Sigma - \Sigma C^T (R + C \Sigma C^T)^{-1} C \Sigma) C^T R^{-1} \tilde{y}, \quad \tilde{y} = y - Cx$$

$$\delta x = \Sigma C^T R^{-1} \tilde{y} - \Sigma C^T (R + C\Sigma C^T)^{-1} C \Sigma C^T R^{-1} \tilde{y}$$

$$= \Sigma C^T (R + C\Sigma C^T)^{-1} (R + C\Sigma C^T - C\Sigma C^T) R^{-1} \tilde{y}$$

$$= \Sigma C^T (R + C\Sigma C^T)^{-1} R R^{-1} \tilde{y}$$

$$= \Sigma C^T (R + C\Sigma C^T)^{-1} \tilde{y}$$

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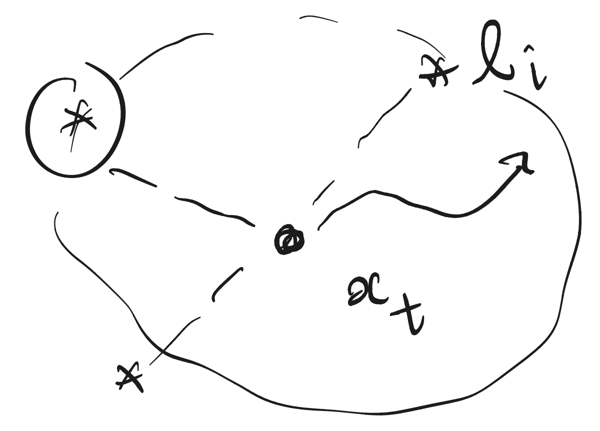
e.g.) Quadrotor SLAM

$$x_{t+1} = A x_t + B u_t + w_t, \quad w \sim N(0, R)$$



$$z_t^i = \| x_t - l_i \|^2 + v_t^i$$