CS5760: Computer Vision

Image alignment

http://www.wired.com/gadgetlab/2010/07/camera-software-lets-you-see-into-the-past/

Reading

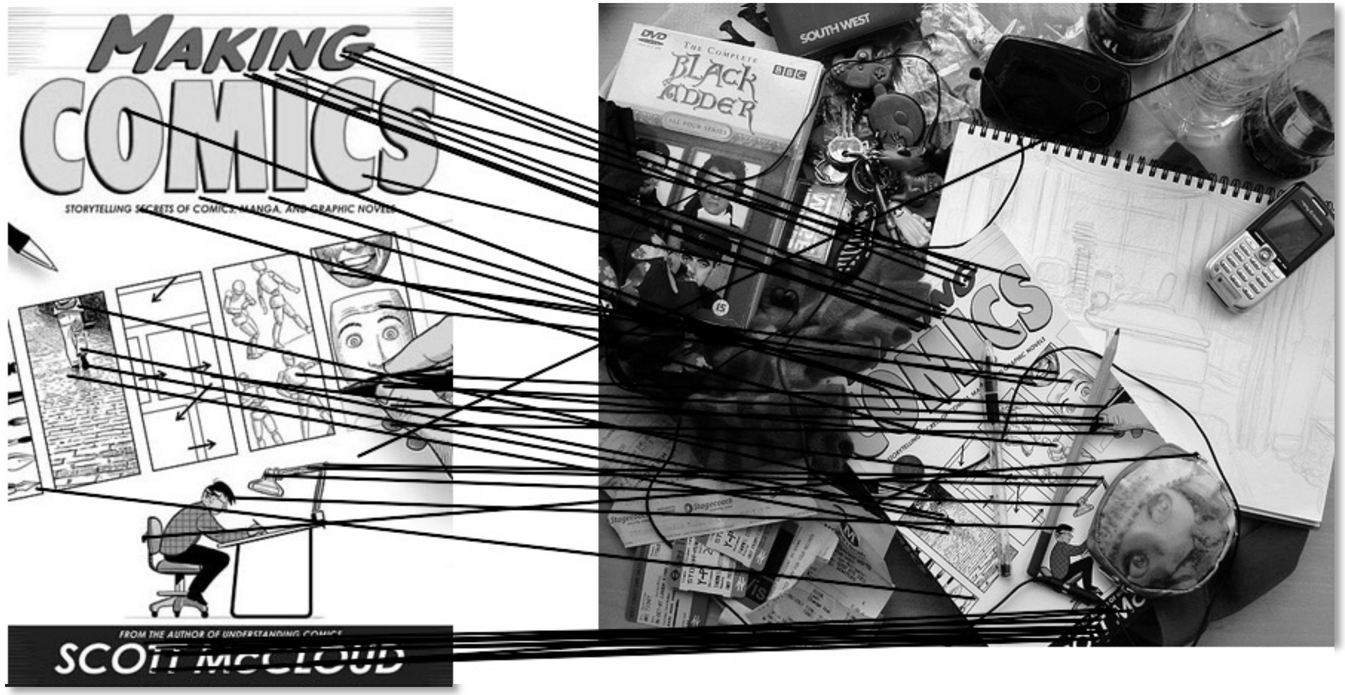
- Szeliski (2nd edition): Chapter 8.1

Announcements

- Project 2 code due this Friday, February 24 by 8pm
  - Please get started now if you haven’t already!
  - Report due **Thursday, March 2** by 8pm on CMSX (note new date)
- Take-home midterm to be released after February Break
  - To be released at 2:15pm Thursday, March 2
  - Due Tuesday, March 7 by 1pm (beginning of class)
  - Open book, open note (but no Google)
  - To be done on your own

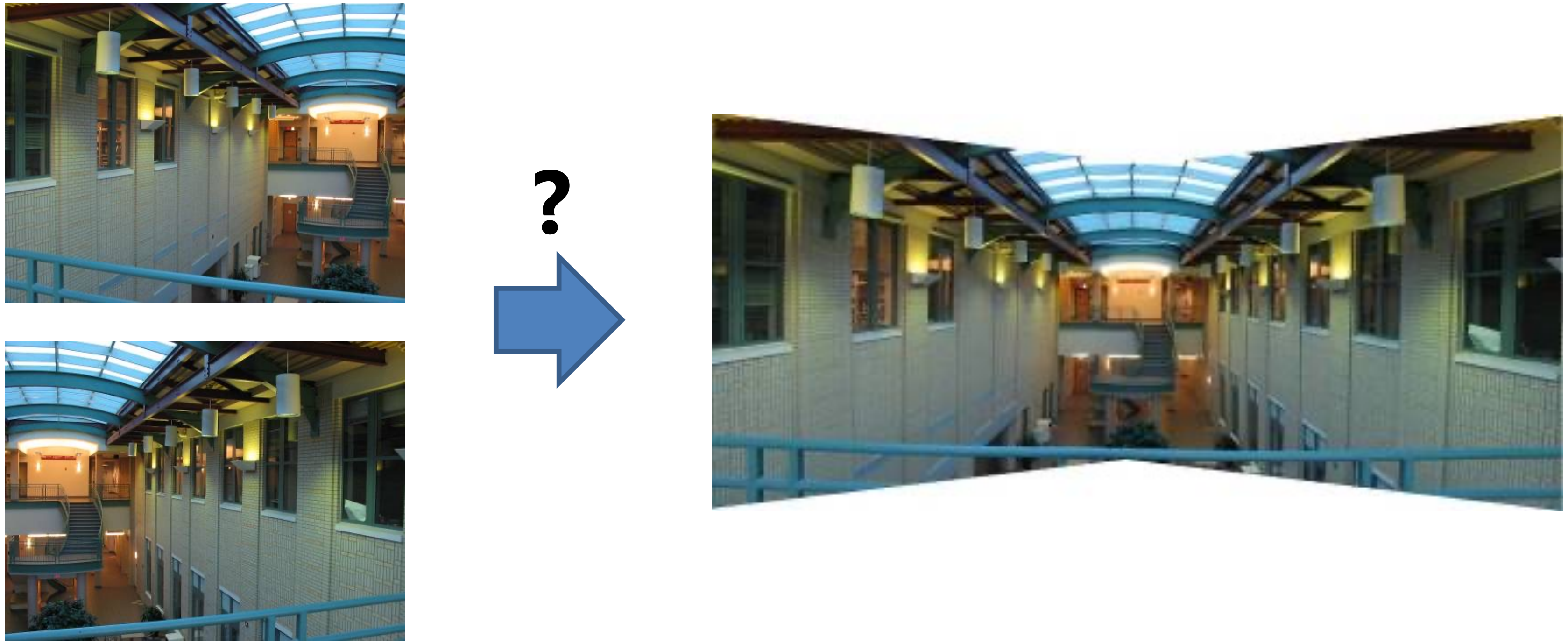
Computing transformations

- Given a set of matches between images A and B
  - How can we compute the transform T from A to B?

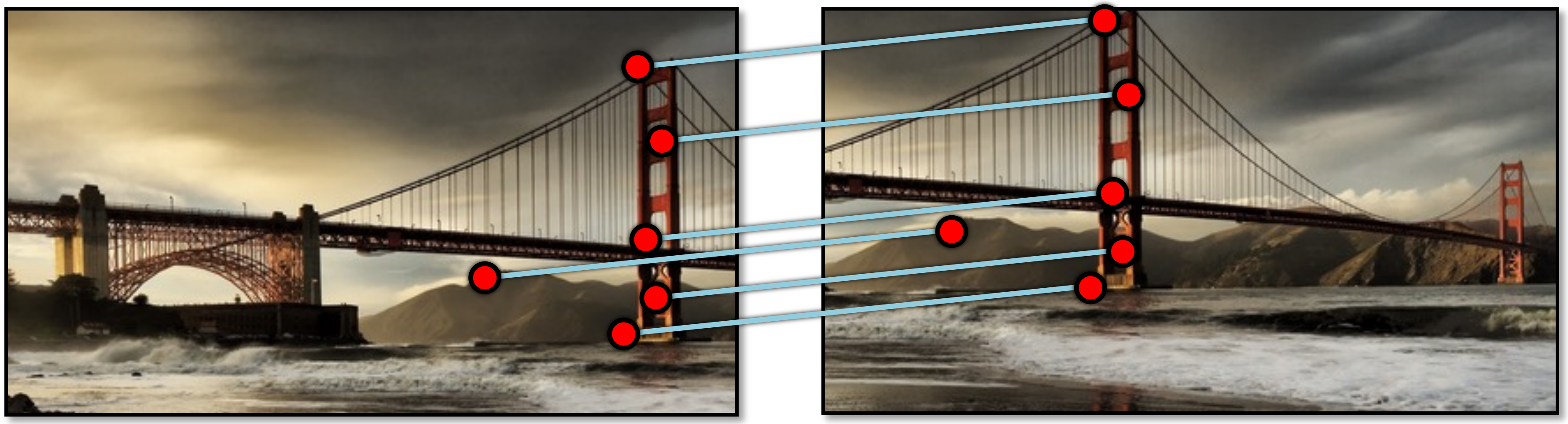


  - Find transform T that best “agrees” with the matches

Computing transformations



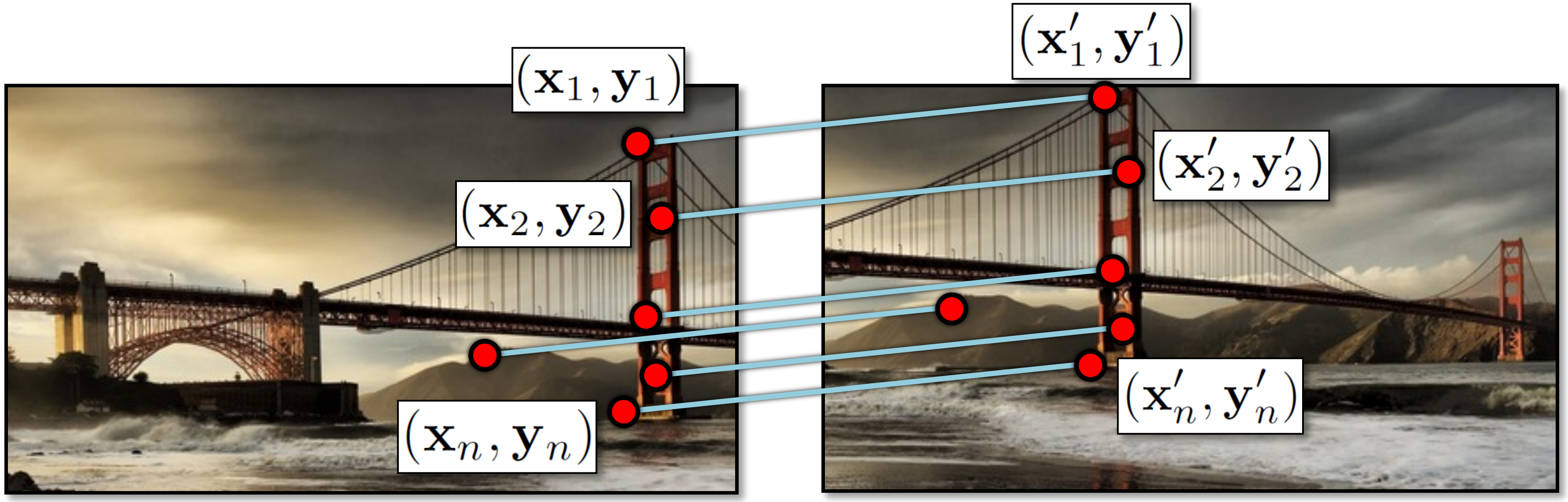
Simple case: translations



$(\mathbf{x}_t, \mathbf{y}_t)$

How do we solve for $(\mathbf{x}_t, \mathbf{y}_t)$ **?**

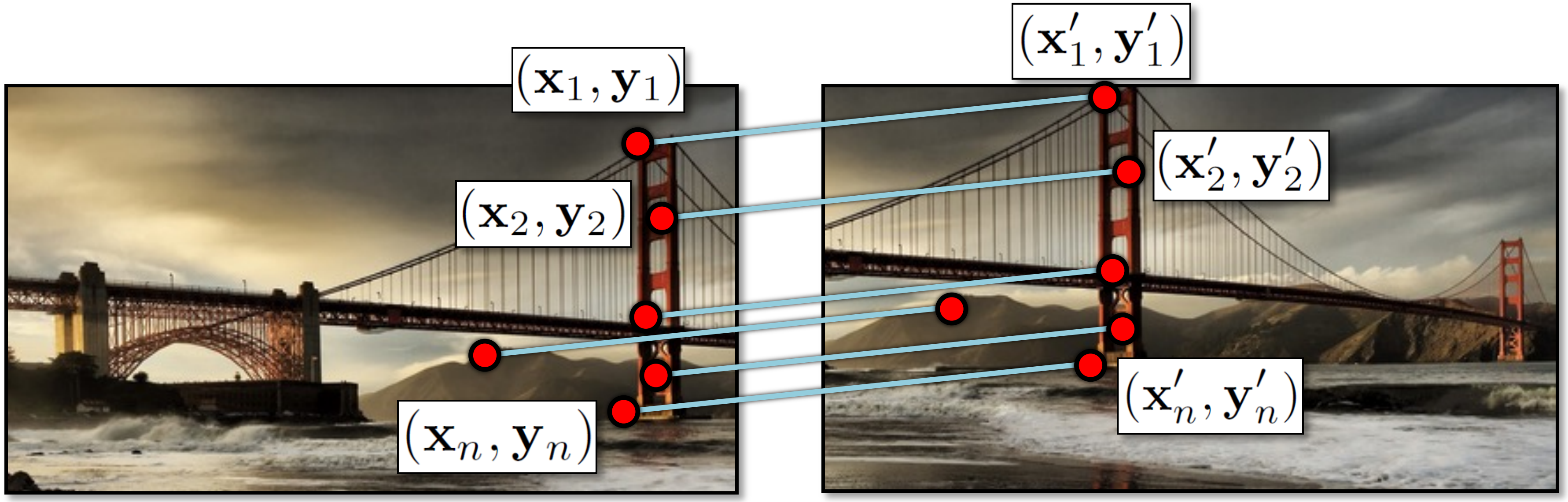
Simple case: translations



Displacement of match i = $(\mathbf{x}'_i - \mathbf{x}_i, \mathbf{y}'_i - \mathbf{y}_i)$

$$(\mathbf{x}_t, \mathbf{y}_t) = \left(\frac{1}{n} \sum_{i=1}^{n} \mathbf{x}'_i - \mathbf{x}_i, \frac{1}{n} \sum_{i=1}^{n} \mathbf{y}'_i - \mathbf{y}_i \right)$$

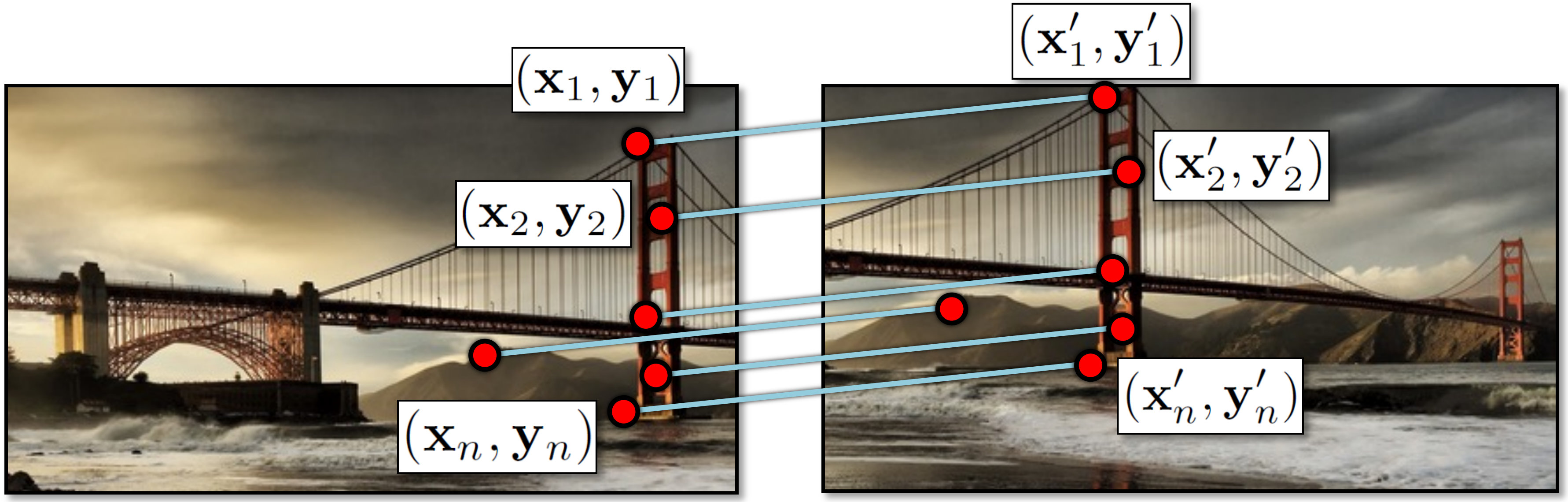
Another view



$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$
$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$

- System of linear equations
  - What are the knowns? Unknowns?
  - How many unknowns? How many equations (per match)?

Another view



$$\mathbf{x}_i + \mathbf{x_t} = \mathbf{x}'_i$$
$$\mathbf{y}_i + \mathbf{y_t} = \mathbf{y}'_i$$

- Problem: more equations than unknowns
  - “Overdetermined” system of equations
  - We will find the *least squares* solution

Least squares formulation

- For each point $(\mathbf{x}_i, \mathbf{y}_i)$

$$\begin{aligned} \mathbf{x}_i + \mathbf{x_t} &= \mathbf{x}'_i \\ \mathbf{y}_i + \mathbf{y_t} &= \mathbf{y}'_i \end{aligned}$$

- we define the *residuals* as

$$\begin{aligned} r_{\mathbf{x}_i}(\mathbf{x}_t) &= (\mathbf{x}_i + \mathbf{x}_t) - \mathbf{x}'_i \\ r_{\mathbf{y}_i}(\mathbf{y}_t) &= (\mathbf{y}_i + \mathbf{y}_t) - \mathbf{y}'_i \end{aligned}$$

Least squares formulation

- Goal: minimize sum of squared residuals

$$C(\mathbf{x}_t, \mathbf{y}_t) = \sum_{i=1}^{n} \left(r_{\mathbf{x}_i}(\mathbf{x}_t)^2 + r_{\mathbf{y}_i}(\mathbf{y}_t)^2 \right)$$

- “Least squares” solution
- For translations, is equal to mean (average) displacement

Least squares formulation

- Can also write as a matrix equation

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \vdots & \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} x'_1 - x_1 \\ y'_1 - y_1 \\ x'_2 - x_2 \\ y'_2 - y_2 \\ \vdots \\ x'_n - x_n \\ y'_n - y_n \end{bmatrix}$$

$$\mathbf{A} \quad \mathbf{t} = \mathbf{b}$$

$2n$ x 2   2 x 1 $2n$ x 1

Least squares

$$\mathbf{A}\mathbf{t} = \mathbf{b}$$

- Find **t** that minimizes

$$||\mathbf{A}\mathbf{t} - \mathbf{b}||^2$$

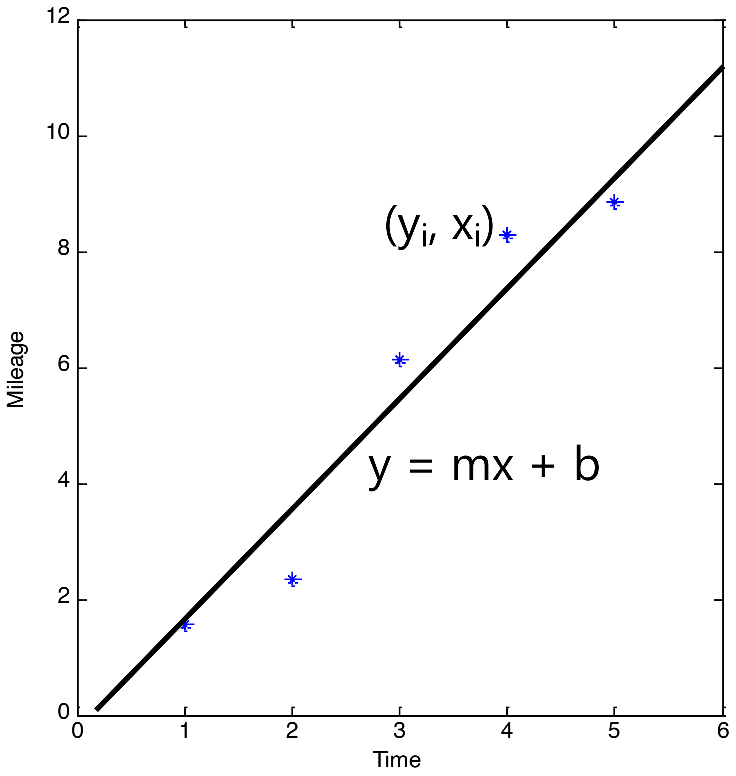
- To solve, form the *normal equations*

$$\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{t} = \mathbf{A}^{\mathrm{T}}\mathbf{b}$$

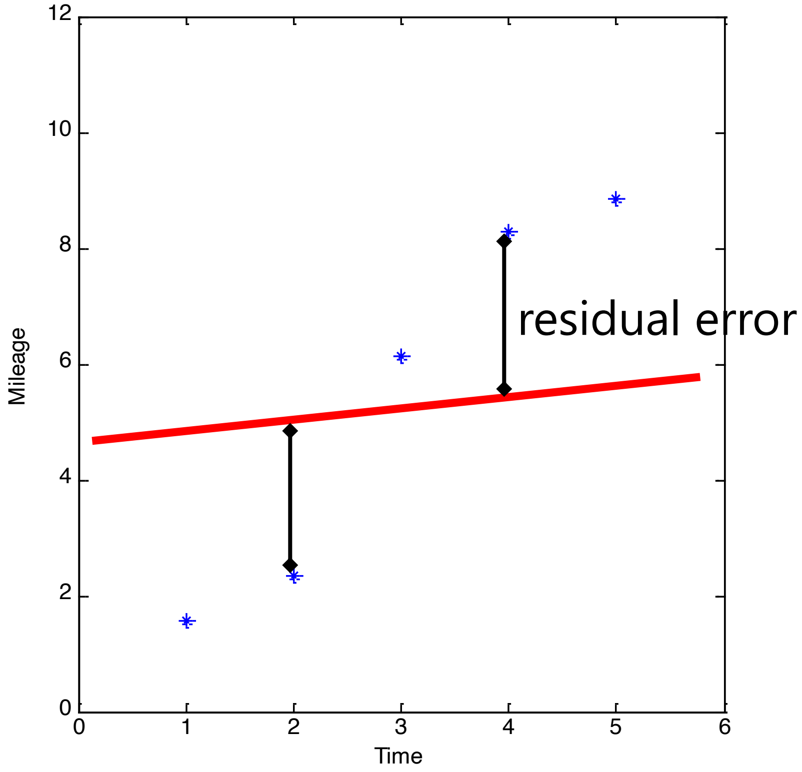
$$\mathbf{t} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b}$$

Questions?

Least squares: linear regression



Linear regression



$$\text{Cost}(m,b) = \sum_{i=1}^{n} |y_i - (mx_i + b)|^2$$

Linear regression

$$\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Affine transformations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- How many unknowns?
- How many equations per match?
- How many matches do we need?

Affine transformations

- Residuals:

$$r_{x_i}(a,b,c,d,e,f) \quad = \quad (ax_i + by_i + c) - x'_i$$
$$r_{y_i}(a,b,c,d,e,f) \quad = \quad (dx_i + ey_i + f) - y'_i$$

- Cost function:

$$C(a,b,c,d,e,f) = \sum_{i=1}^{n} \left(r_{x_i}(a,b,c,d,e,f)^2 + r_{y_i}(a,b,c,d,e,f)^2 \right)$$

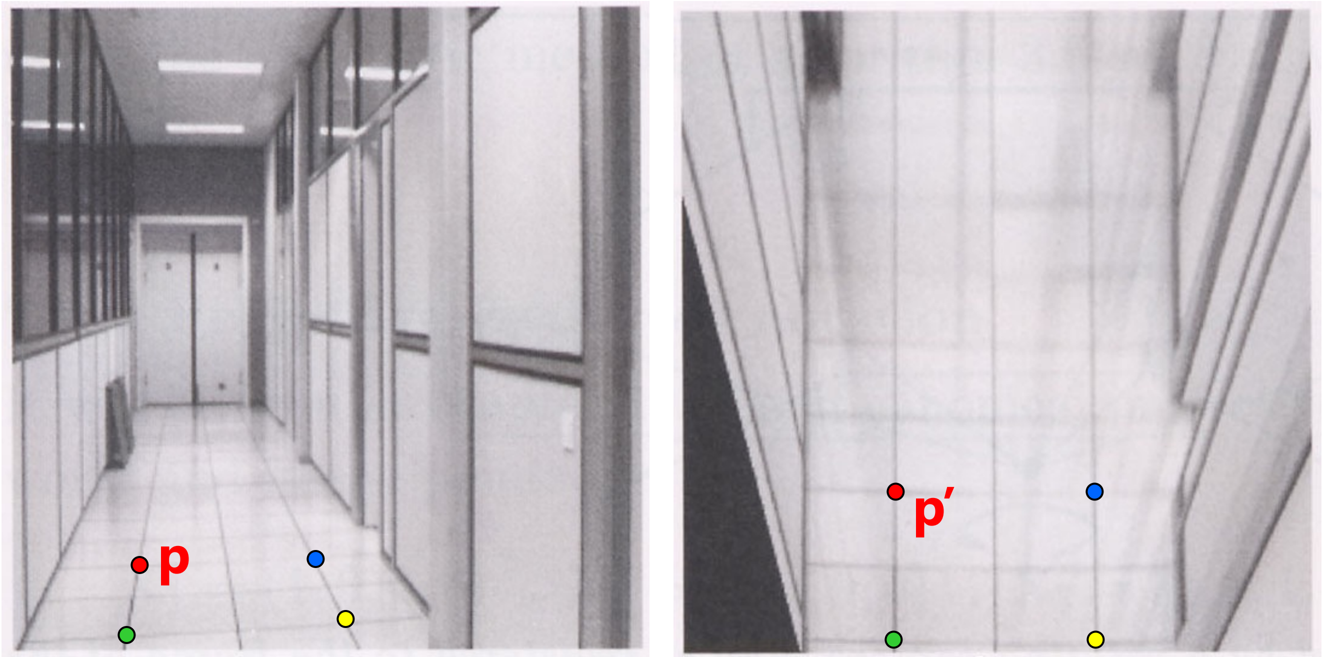
Affine transformations

- Matrix form

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ & & & \vdots & & \\ x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} x'_1 \\ y'_1 \\ x'_2 \\ y'_2 \\ \vdots \\ x'_n \\ y'_n \end{bmatrix}$$

$$\underset{2n \times 6}{\mathbf{A}} \quad \underset{6 \times 1}{\mathbf{t}} = \underset{2n \times 1}{\mathbf{b}}$$

Homographies



To unwarp (rectify) an image

- solve for homography **H** given **p** and **p′**
- solve equations of the form: w**p′** = **Hp**
  - linear in unknowns: w and coefficients of **H**
  - H is defined up to an arbitrary scale factor
  - how many matches are necessary to solve for **H**?

Solving for homographies

$$\begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}$$

$$x'_i = \frac{h_{00}x_i + h_{01}y_i + h_{02}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

$$y'_i = \frac{h_{10}x_i + h_{11}y_i + h_{12}}{h_{20}x_i + h_{21}y_i + h_{22}}$$

Not linear!

$$x'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{00}x_i + h_{01}y_i + h_{02}$$
$$y'_i(h_{20}x_i + h_{21}y_i + h_{22}) = h_{10}x_i + h_{11}y_i + h_{12}$$

Solving for homographies

$$\begin{aligned} x'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= h_{00}x_i + h_{01}y_i + h_{02} \\ y'_i(h_{20}x_i + h_{21}y_i + h_{22}) &= h_{10}x_i + h_{11}y_i + h_{12} \end{aligned}$$

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -x'_i x_i & -x'_i y_i & -x'_i \\ 0 & 0 & 0 & x_i & y_i & 1 & -y'_i x_i & -y'_i y_i & -y'_i \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Solving for homographies

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -x'_1 x_1 & -x'_1 y_1 & -x'_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -y'_1 x_1 & -y'_1 y_1 & -y'_1 \\ & & & & \vdots & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & -x'_n x_n & -x'_n y_n & -x'_n \\ 0 & 0 & 0 & x_n & y_n & 1 & -y'_n x_n & -y'_n y_n & -y'_n \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{2n \times 9}{\mathbf{A}} \quad \underset{9}{\mathbf{h}} \quad \underset{2n}{\mathbf{0}}$$

Defines a least squares problem: minimize $\|\mathbf{A}\mathbf{h} - \mathbf{0}\|^2$

- Since $\mathbf{h}$ is only defined up to scale, solve for unit vector $\hat{\mathbf{h}}$
- Solution: $\hat{\mathbf{h}}$ = eigenvector of $\mathbf{A}^T\mathbf{A}$ with smallest eigenvalue
- Works with 4 or more points

Recap: Two Common Optimization Problems

Problem statement

minimize $\|\mathbf{Ax} - \mathbf{b}\|^2$

(least squares solution to $\mathbf{Ax} = \mathbf{b}$)

Solution

$\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$

$\mathbf{x} = \mathbf{A}\backslash\mathbf{b}$ (matlab)

Problem statement

minimize $\mathbf{x}^T\mathbf{A}^T\mathbf{Ax}$ s.t. $\mathbf{x}^T\mathbf{x} = 1$

(non−trivial lsq solution to $\mathbf{Ax} = 0$)

Solution

$[\mathbf{v}, \lambda] = \mathrm{eig}(\mathbf{A}^T\mathbf{A})$

$\lambda_1 < \lambda_{2..n}: \mathbf{x} = \mathbf{v}_1$

Computing transformations

Questions?

Image alignment algorithm

Given images A and B

1. Compute image features for A and B
2. Match features between A and B
3. Compute homography between A and B using least squares on set of matches

What could go wrong?

Outliers