CS**5643**

09 Rigid body motion

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Overview

Kinematics of rigid bodies (emphasis on 2D case)

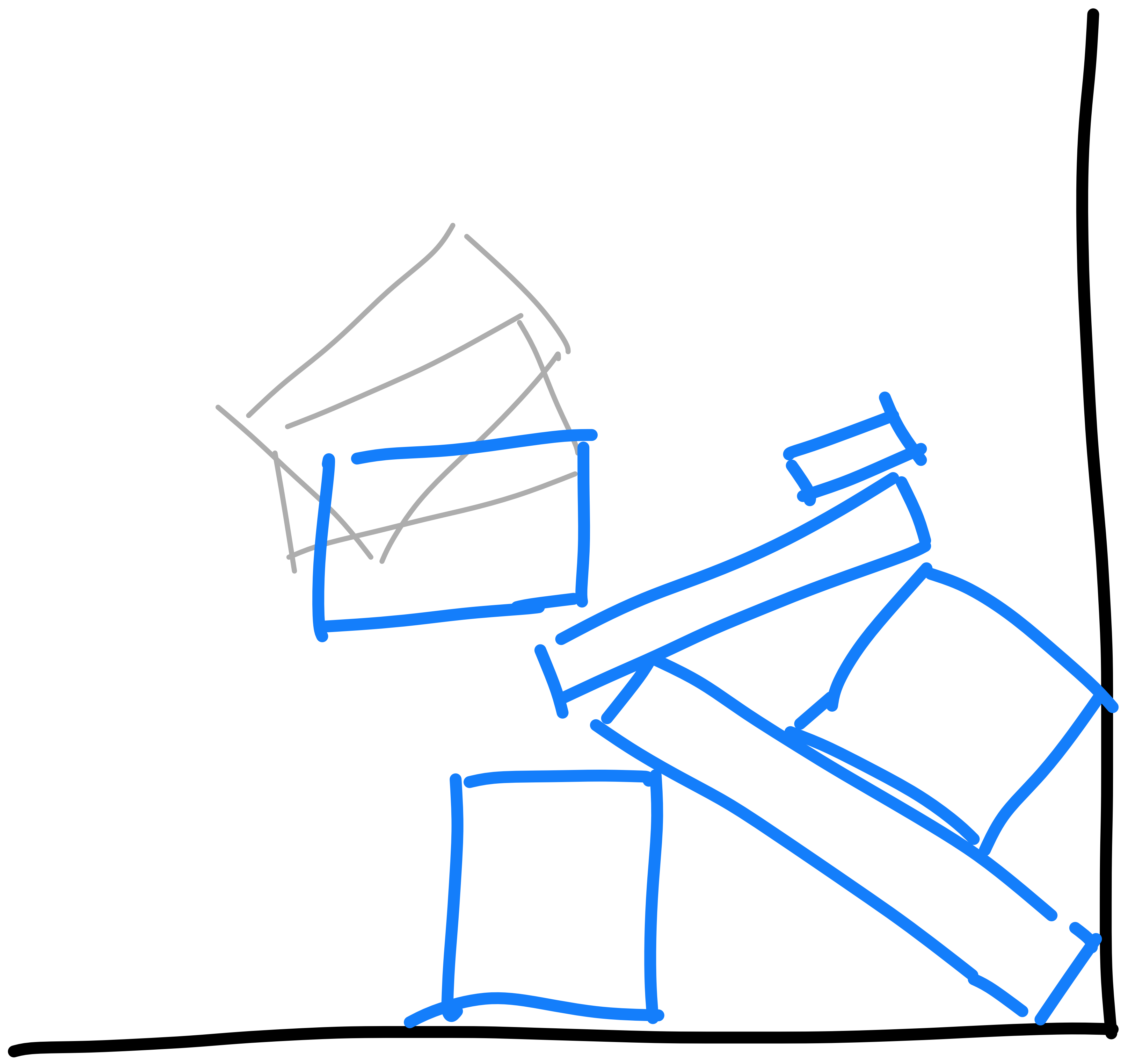
- state includes position and rotation for each body

Dynamics of a free body

- how to compute time derivative of state
- forces, torques, impulses

Rigid body collisions

- isolated body-obstacle collision
- isolated body-body collision



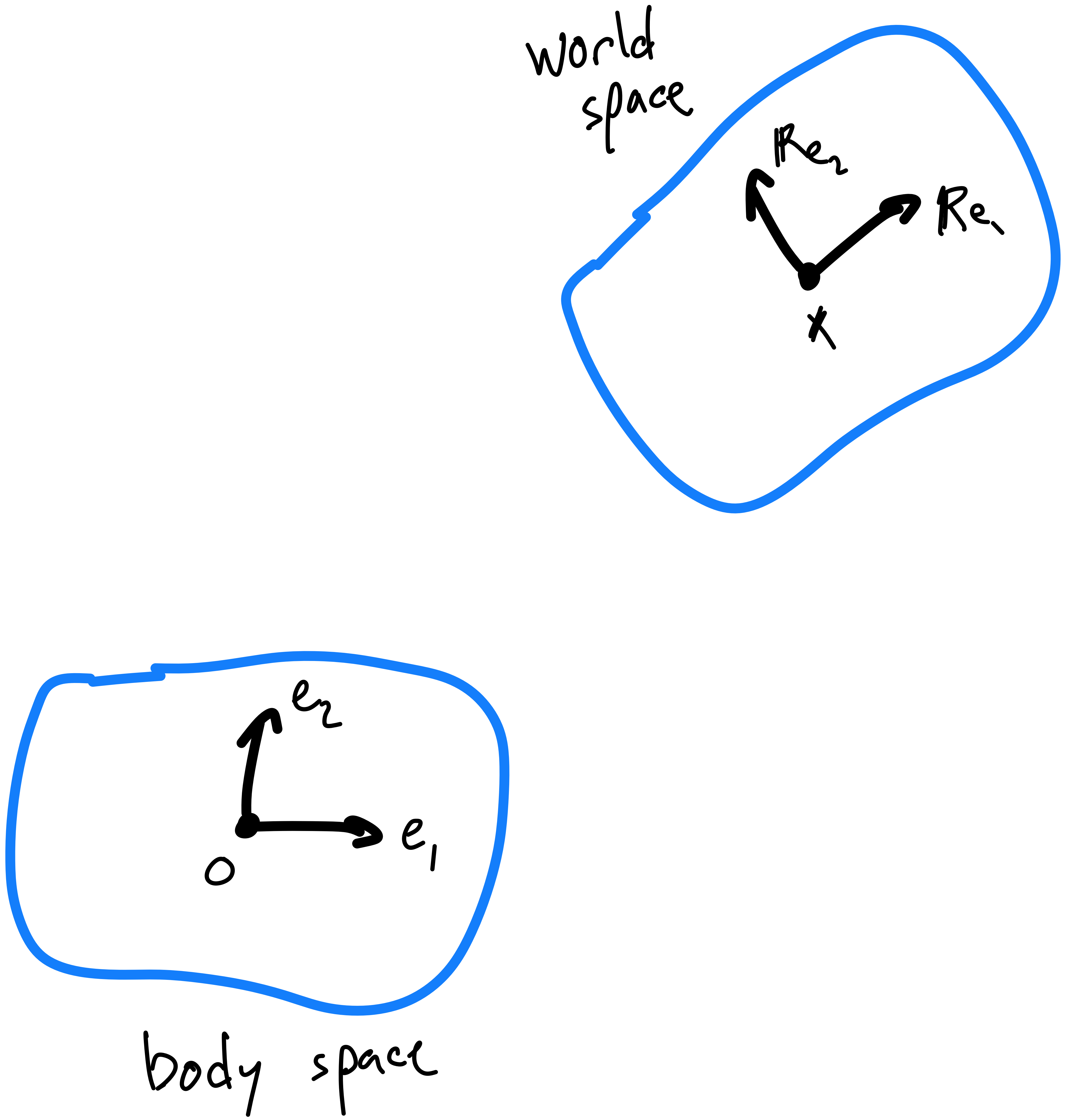
Rigid body state

A position

- I'll call it $\mathbf{x}$
- it's the position of the center of mass (keeps things simpler)

A rotation

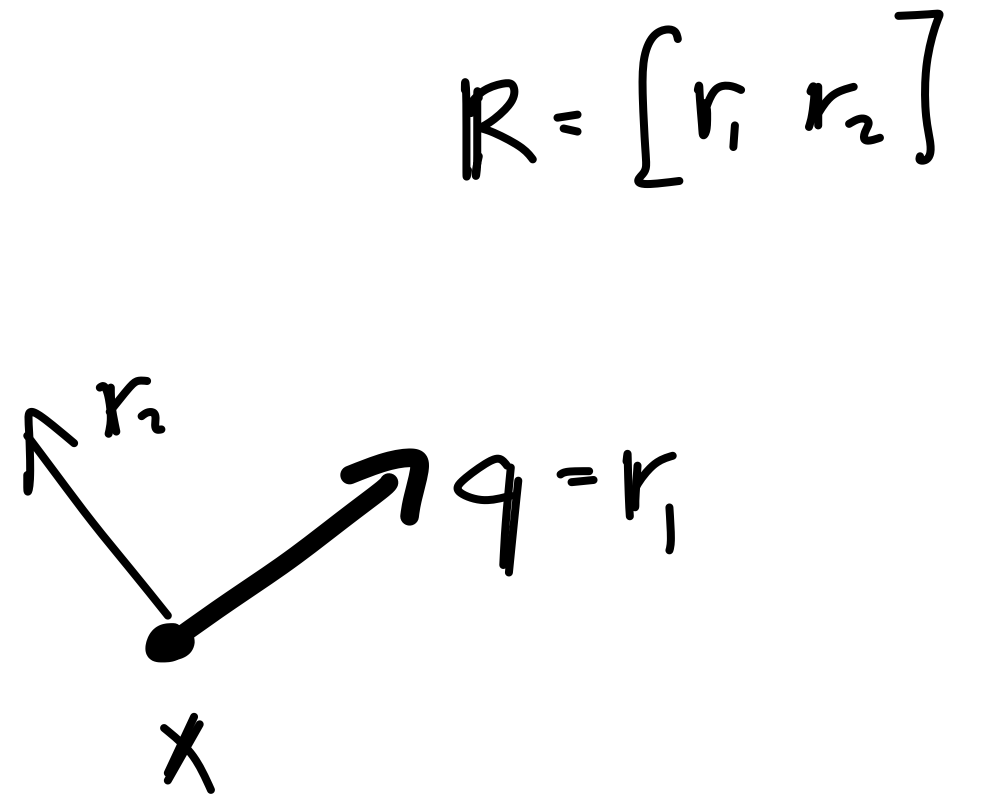
- can be represented with a rotation matrix $\mathbf{R}$
- defines the mapping from the body's local space to world space:
  - $\mathbf{r} = \mathbf{x} + \mathbf{R}\mathbf{r}_b$ — $\mathbf{r}$ is in world space, $\mathbf{r}_b$ is in body space



Representing rigid body state

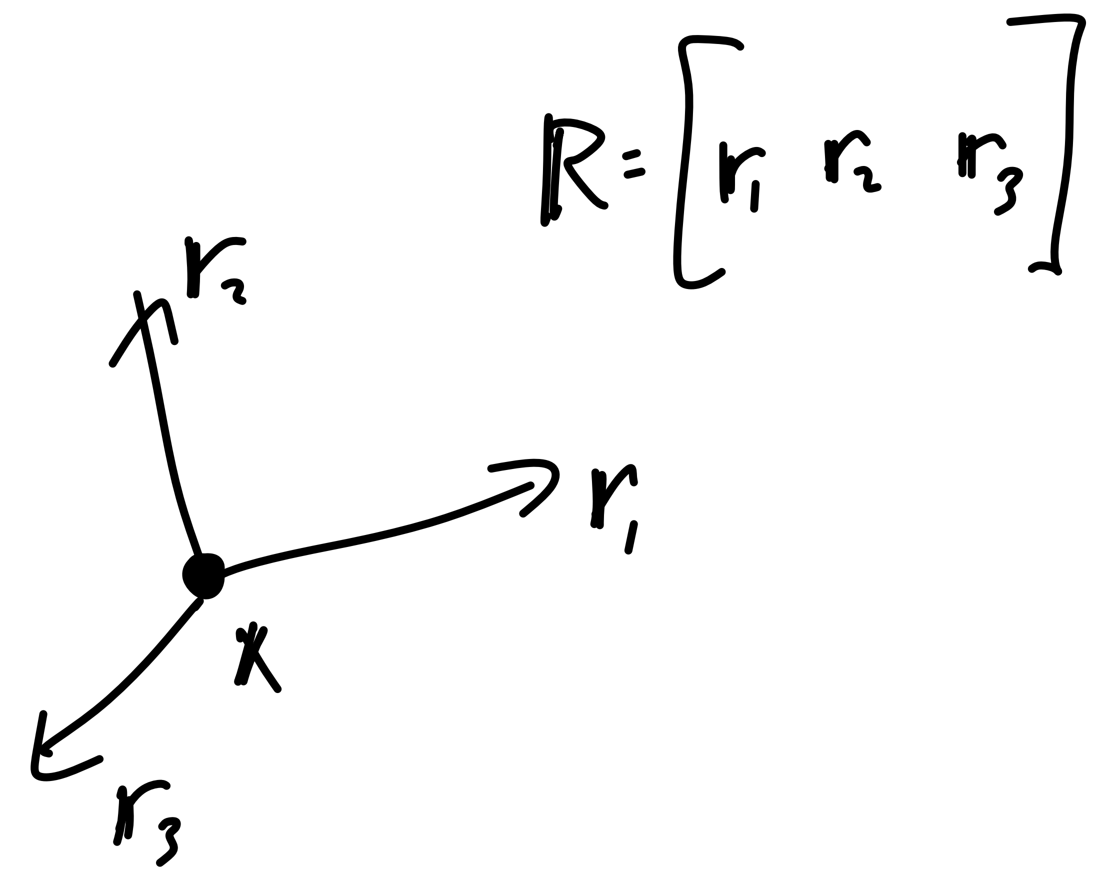
In 2D

- $\mathbf{x}$ is simple (2 numbers)
- $\mathbf{R} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ so let's write down $\mathbf{q} = \begin{bmatrix} c \\ s \end{bmatrix}$
- so state has 4 numbers but 3 DoF since $\|\mathbf{q}\| = 1$



In 3D

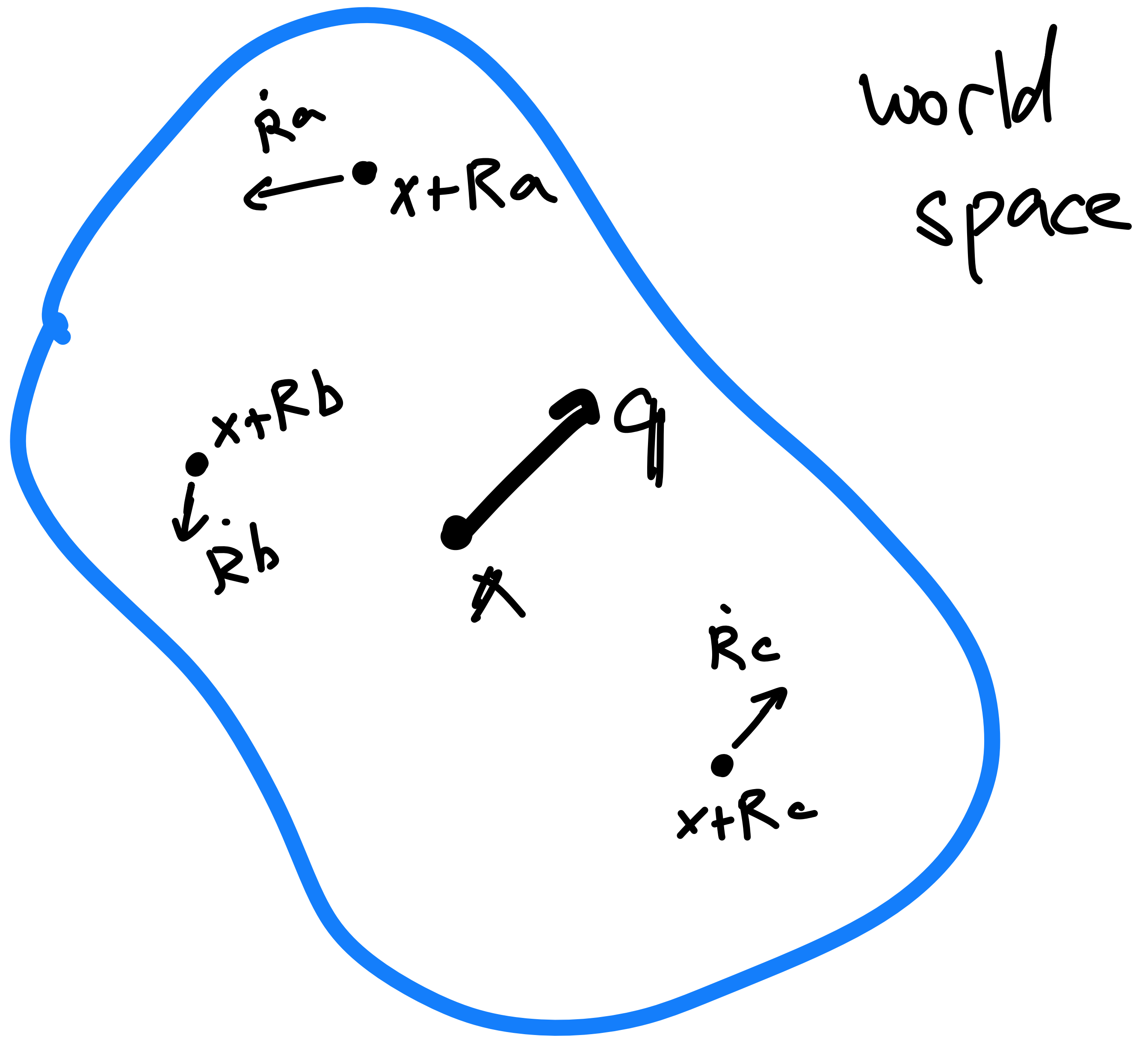
- $\mathbf{x}$ is simple (3 numbers)
- rotation is best represented as a unit quaternion
  - $\mathbf{q} = [w \quad x \quad y \quad z]^T$
  - $\mathbf{R}(t) = \mathbf{R}(\mathbf{q}(t))$
- so state has 7 DoF but 6 DoF since $\|\mathbf{q}\| = 1$

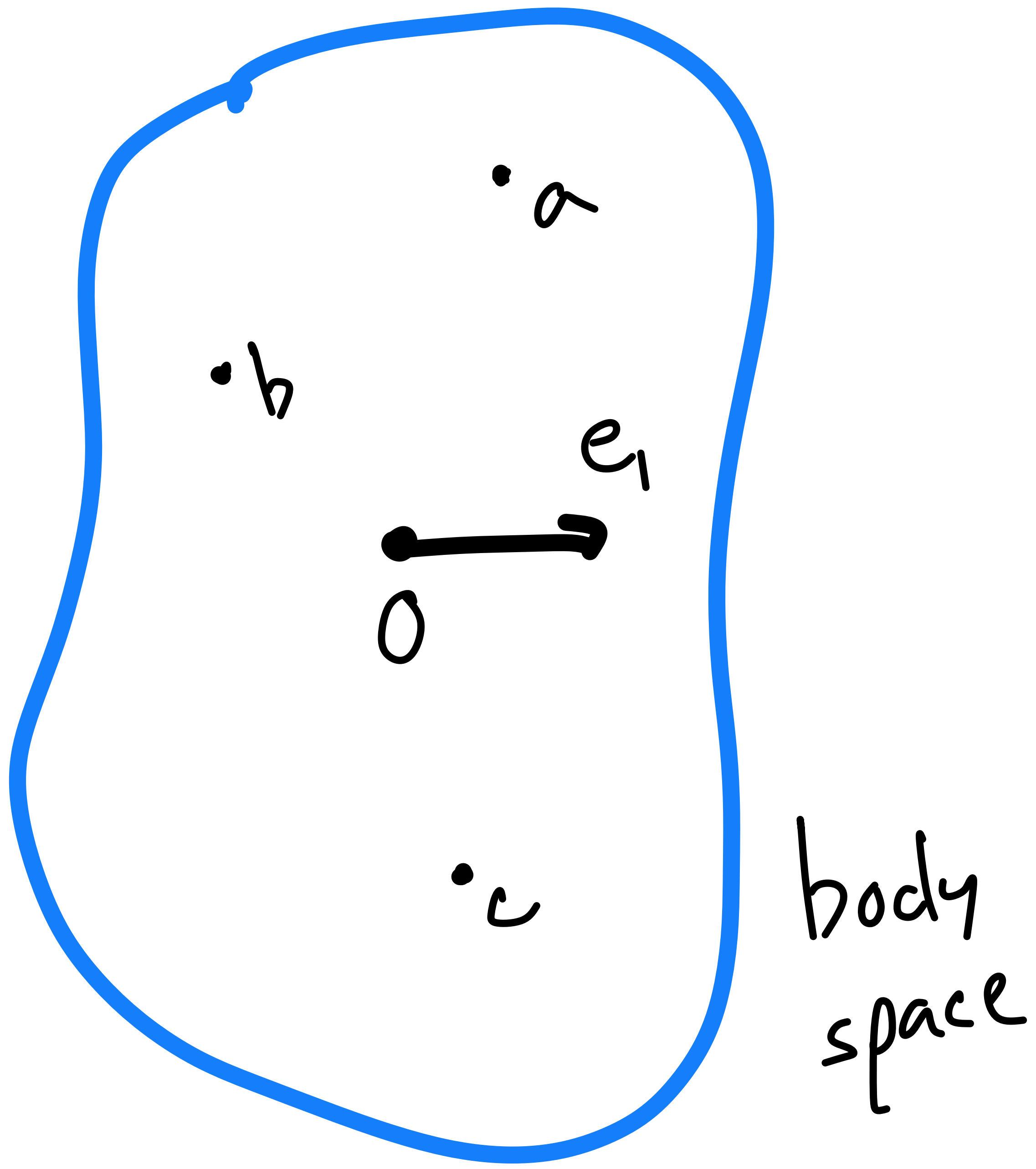


Rigid body velocity

Motion of a point on a moving body

- $\mathbf{r}(t) = \mathbf{x} + \mathbf{R}(t)\mathbf{r}_b$ ($\mathbf{r}_b$ is not changing)
- $\dot{\mathbf{r}}(t) = \dot{\mathbf{r}}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_b = \mathbf{v}(t) + \dot{\mathbf{R}}(t)\mathbf{r}_b$
- so $\dot{\mathbf{R}}$ maps a body-space point to the rotational part of its world-space velocity

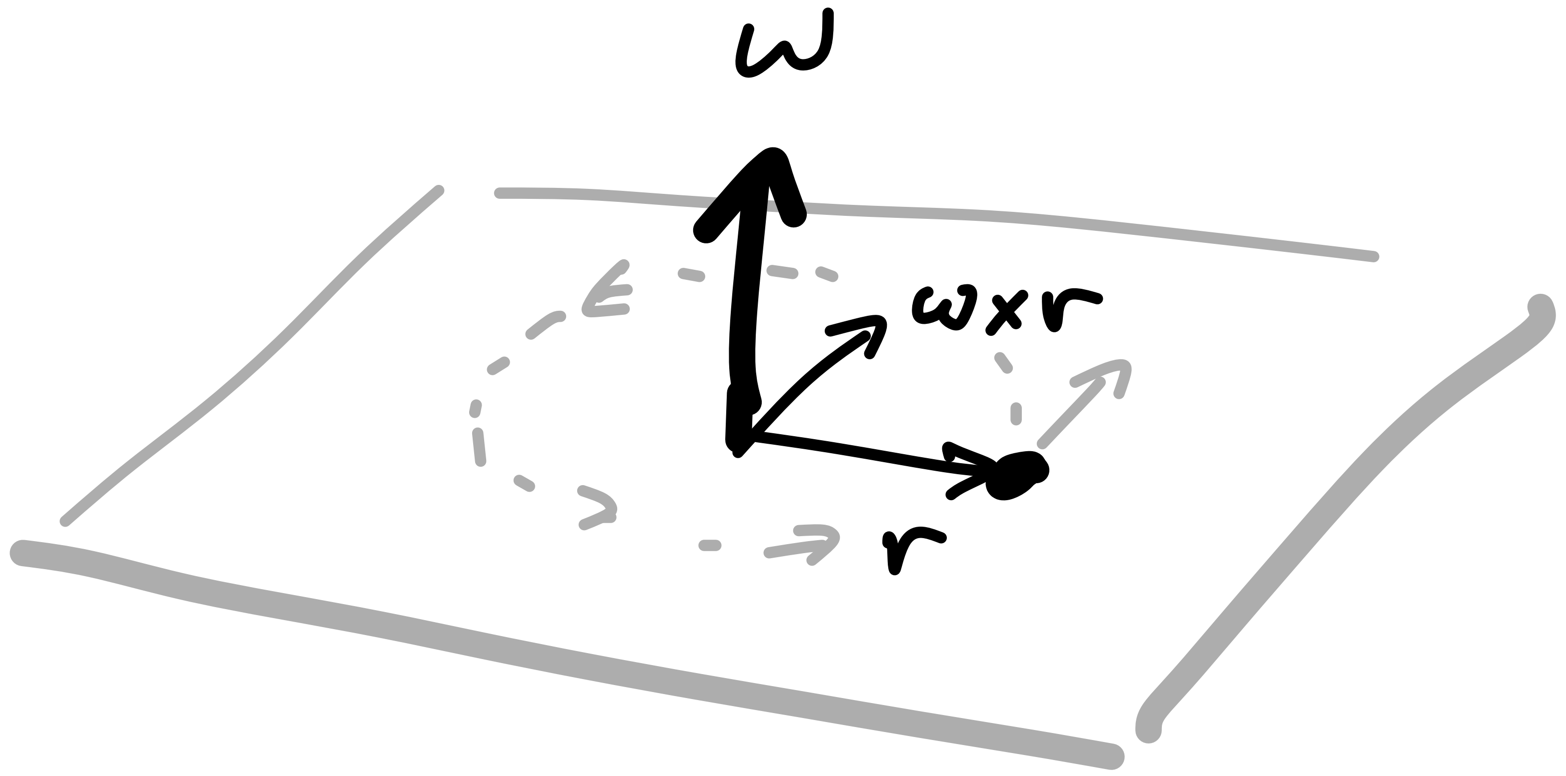




Angular velocity in 2D

The matrix $\dot{\mathbf{R}}$ is special, just like $\mathbf{R}$

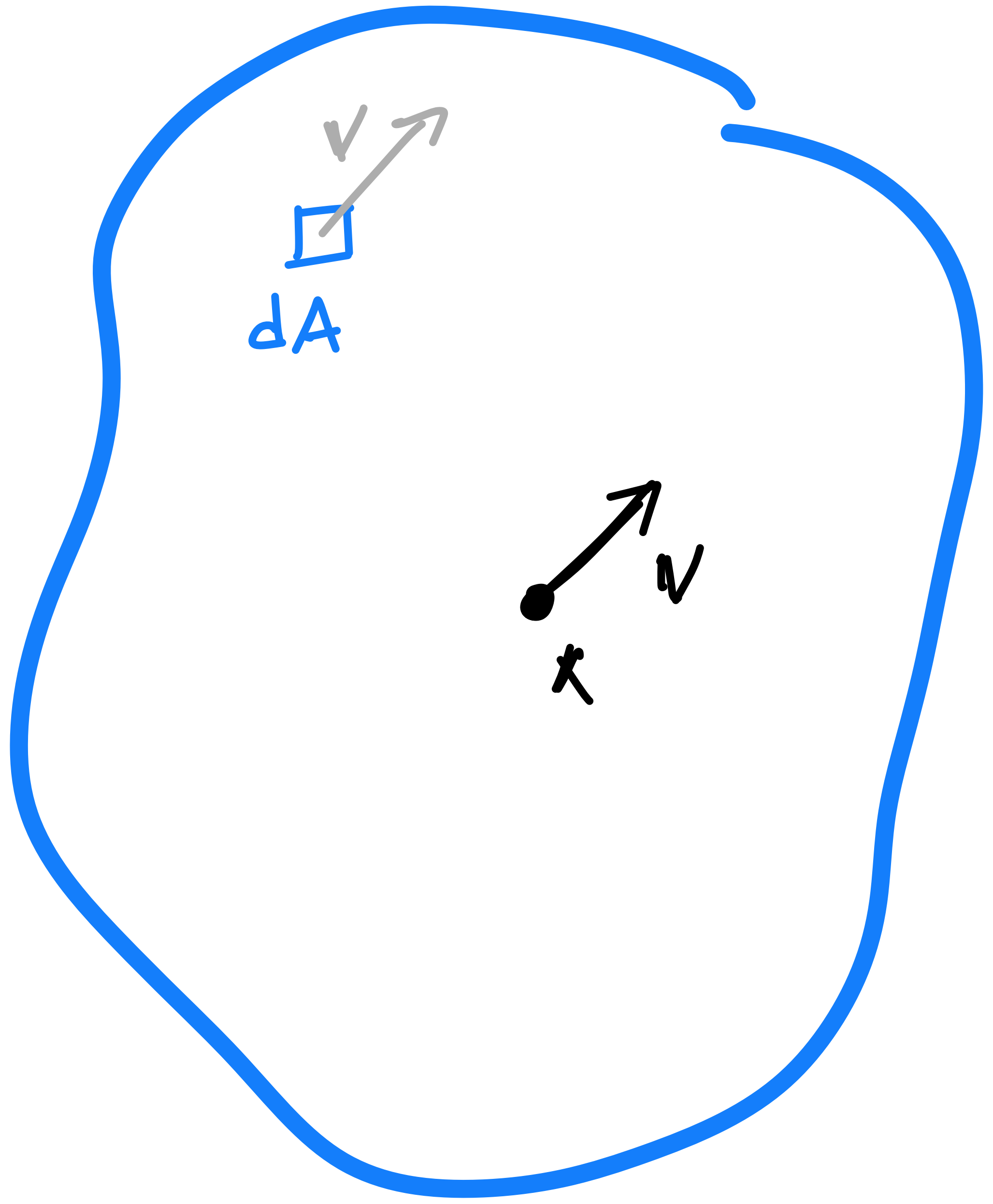
- we also don't need to write down the whole matrix
- look at 2D case with a steady rotation
- conclusion: $\dot{\mathbf{R}} = \omega^\times \mathbf{R}$ where $\omega^\times = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$
- ω is called the angular velocity
- since $\mathbf{q}$ is the first column of $\mathbf{R}$, $\dot{\mathbf{q}} = \omega^\times \mathbf{q} = \omega \times \mathbf{q}$



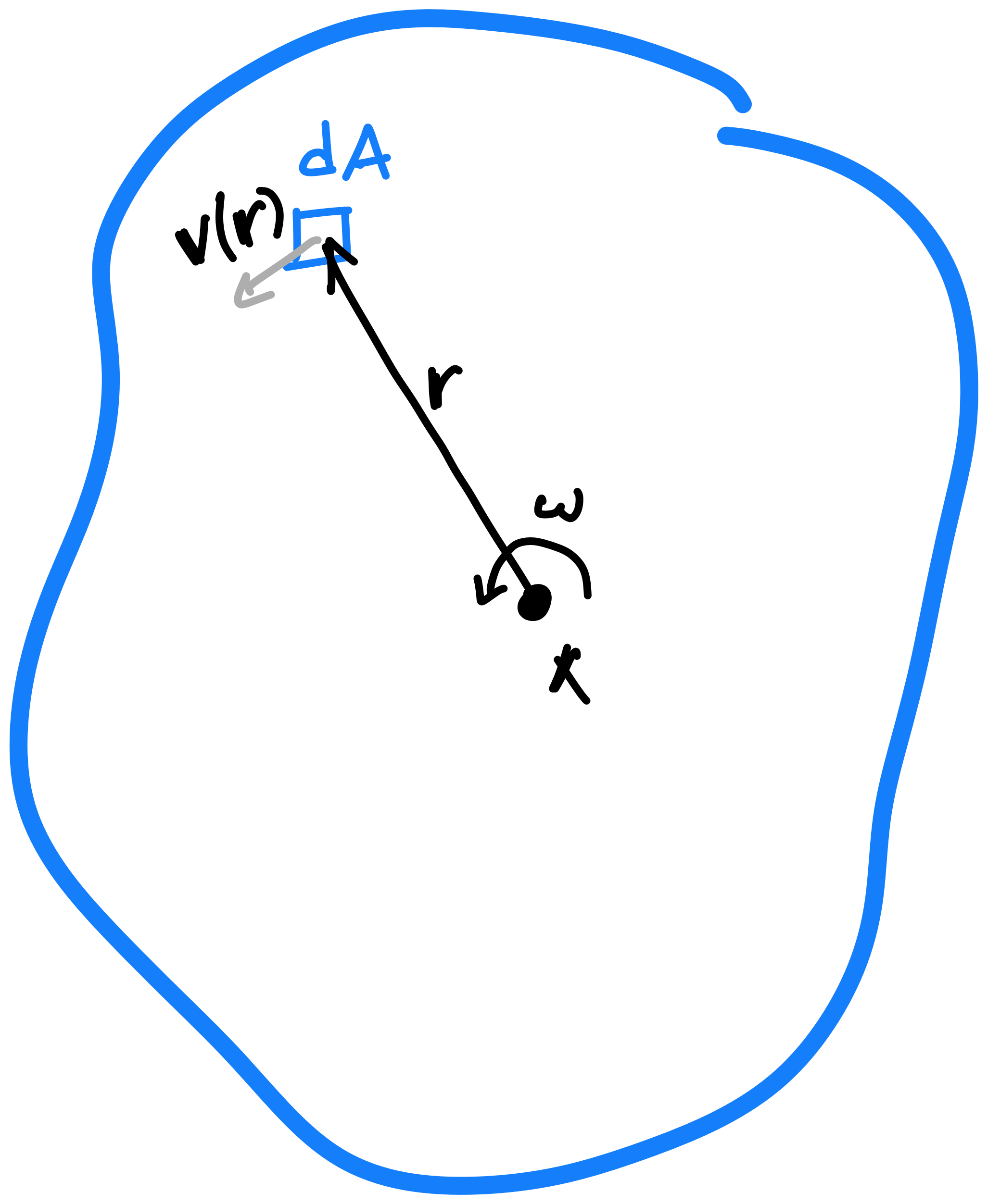
Rigid body kinetic energy (2D)

What is the kinetic energy of a body with velocity $\mathbf{v}$?

- can arrive at this by integrating kinetic energy density over the body
  - let $\rho(\mathbf{r})$ be the mass density of the body (mass per unit area, in 2D, $\mathbf{r}$ in body coords)
  - differential area dA has velocity $\mathbf{v}$ and kinetic energy $\frac{1}{2}\rho(\mathbf{r})\mathbf{v}^2(\mathbf{r})\,dA$
  - integrate over the body to get $\frac{1}{2}m\mathbf{v}^2$ where $m = \int \rho(\mathbf{r})\,dA$
  - $E_k^{\text{tr}} = \frac{1}{2}m\mathbf{v}^2 = \frac{1}{2}\mathbf{v} \cdot m\mathbf{v} = \frac{1}{2}\mathbf{v} \cdot \mathbf{p}$ — where $\mathbf{p}$ is momentum



Rigid body kinetic energy (2D)



What is the kinetic energy of a body with angular velocity ω?

- apply same to rotating body to get rotational kinetic energy
  - differential area dA at $\mathbf{r}$ has velocity $\omega||\mathbf{r}||$ and kinetic energy $\frac{1}{2}\rho(\mathbf{r})\omega^2\mathbf{r}^2\,dA$
  - integrate over the body to get $\frac{1}{2}I\omega^2$ where $I = \int \mathbf{r}^2\rho(\mathbf{r})\,dA$
  - $E_k^{\text{rot}} = \frac{1}{2}I\omega^2 = \frac{1}{2}\omega \cdot I\omega = \frac{1}{2}\omega \cdot L$ — where L is *angular momentum*

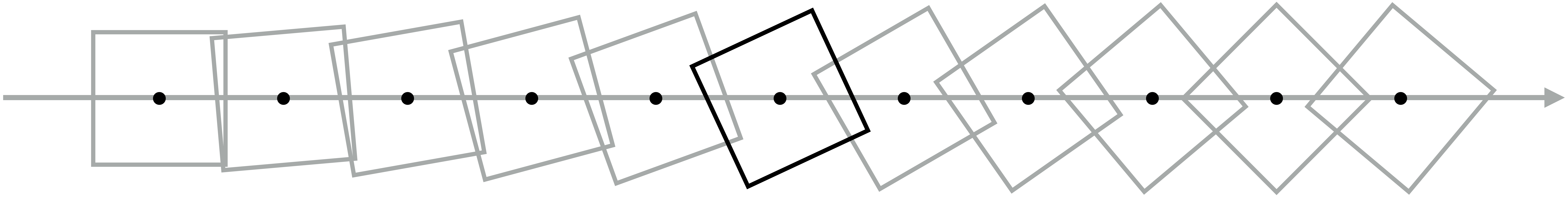
What is this I?

- total body mass weighted by squared distance from origin
- measures how much energy is needed to get the body spinning
- value depends on center; but remember we standardized on having the body origin at the center of mass: $\mathbf{r}_c = \frac{1}{m}\int \mathbf{r}\rho(\mathbf{r})\,dA = \mathbf{0}$ in body coordinates

Rigid body kinetic energy (2D)

What is the kinetic energy of a body with velocity $\mathbf{v}$ and angular velocity ω?

- remember our body origin is at the center of mass
- in this case just add the two energies together: $E_k = \frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}I\omega^2$

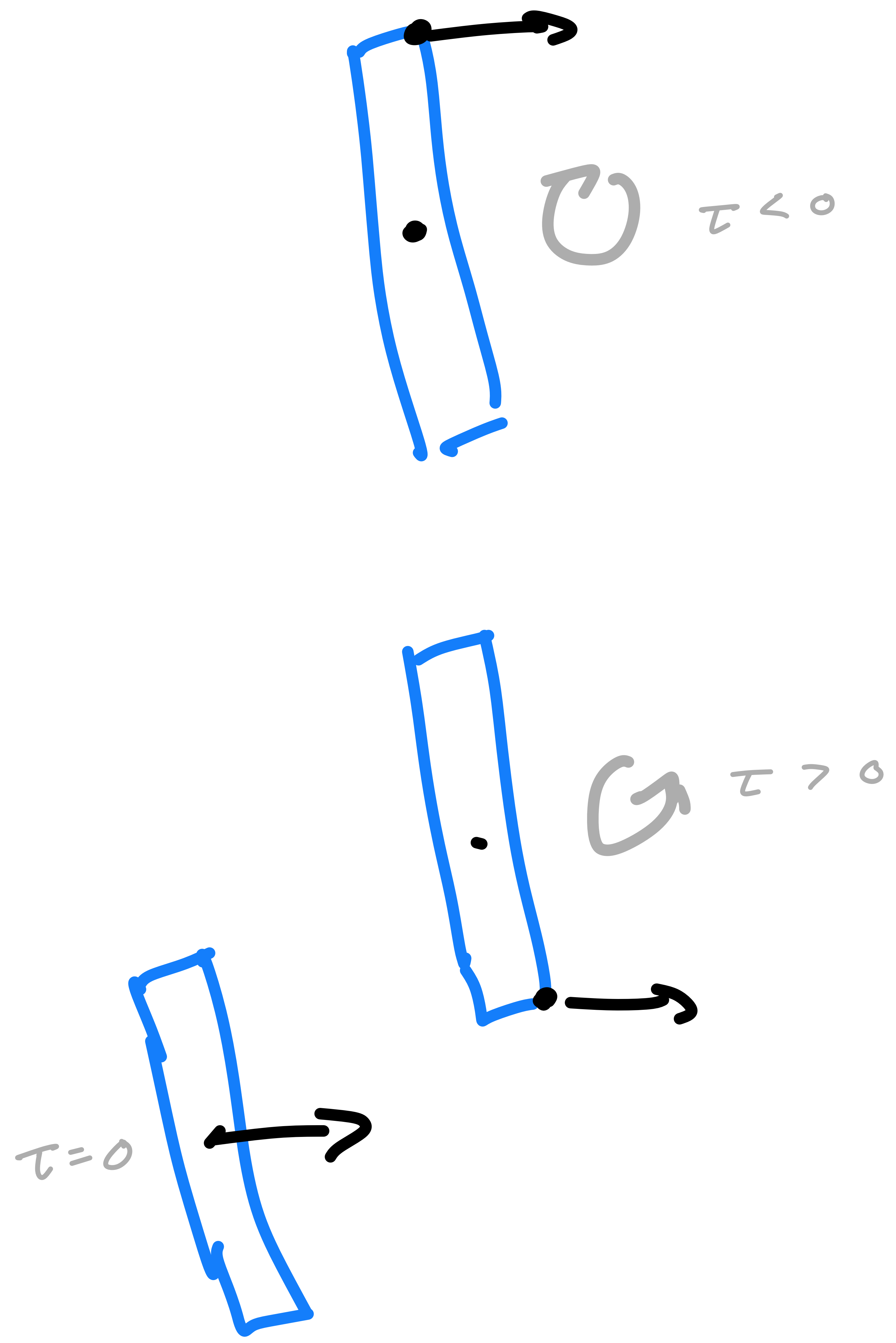


for testing…

Forces and torques

When a force is applied to a point $\mathbf{r}$ on a body

- the force affects the center-of-mass velocity
  - $\mathbf{f} = m\dot{\mathbf{v}} = \dot{\mathbf{p}}$
- the force also affects the angular velocity
  - effect depends on offset $\mathbf{r}' = \mathbf{r} - \mathbf{x}$
  - only the component perpendicular to $\mathbf{r}'$ affects the body's rotation
  - effect is proportional to $||\mathbf{r}'||$
  - hence define torque $\tau = \mathbf{r}' \times \mathbf{f}$
  - $\tau = I\dot{\omega} = \dot{L}$
  - L is constant in the absence of torques



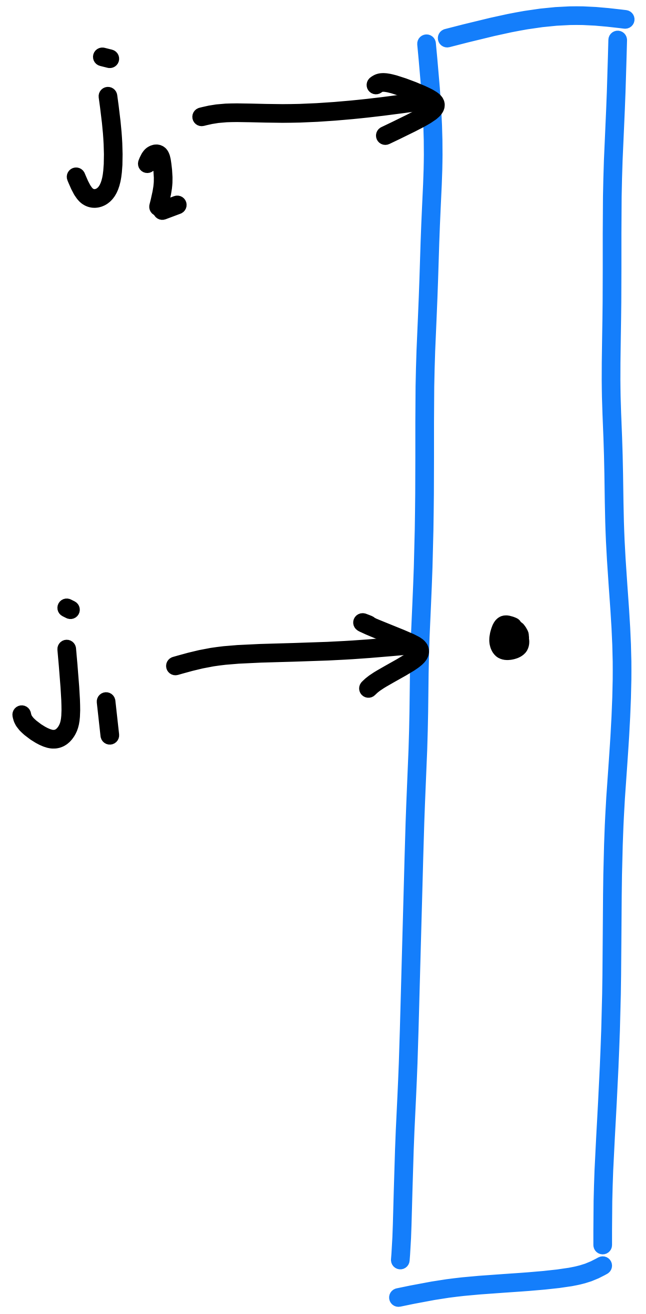
Impulses

Just like with particles, impulses cause instantaneous change in velocity

- for linear velocity, $m\Delta\mathbf{v} = \mathbf{j}$ just like with a particle
- and for angular velocity, $I\Delta\omega = \mathbf{r}' \times \mathbf{j}$ (a torque impulse)

This will be useful for collisions

- $\mathbf{v}^+ = \mathbf{v}^- + m^{-1}\mathbf{j}$
- $\omega^+ = \omega^- + I^{-1}\,\mathbf{r}' \times \mathbf{j}$



POLL

bar with length $l = 4$, mass $m = 3$ and $I = 4$ starts with $\mathbf{v} = \mathbf{0}$ and $\omega = 0$

impulse $\mathbf{j} = (1{,}0)$ is applied (1) at the center of the bar or (2) at the end of the bar

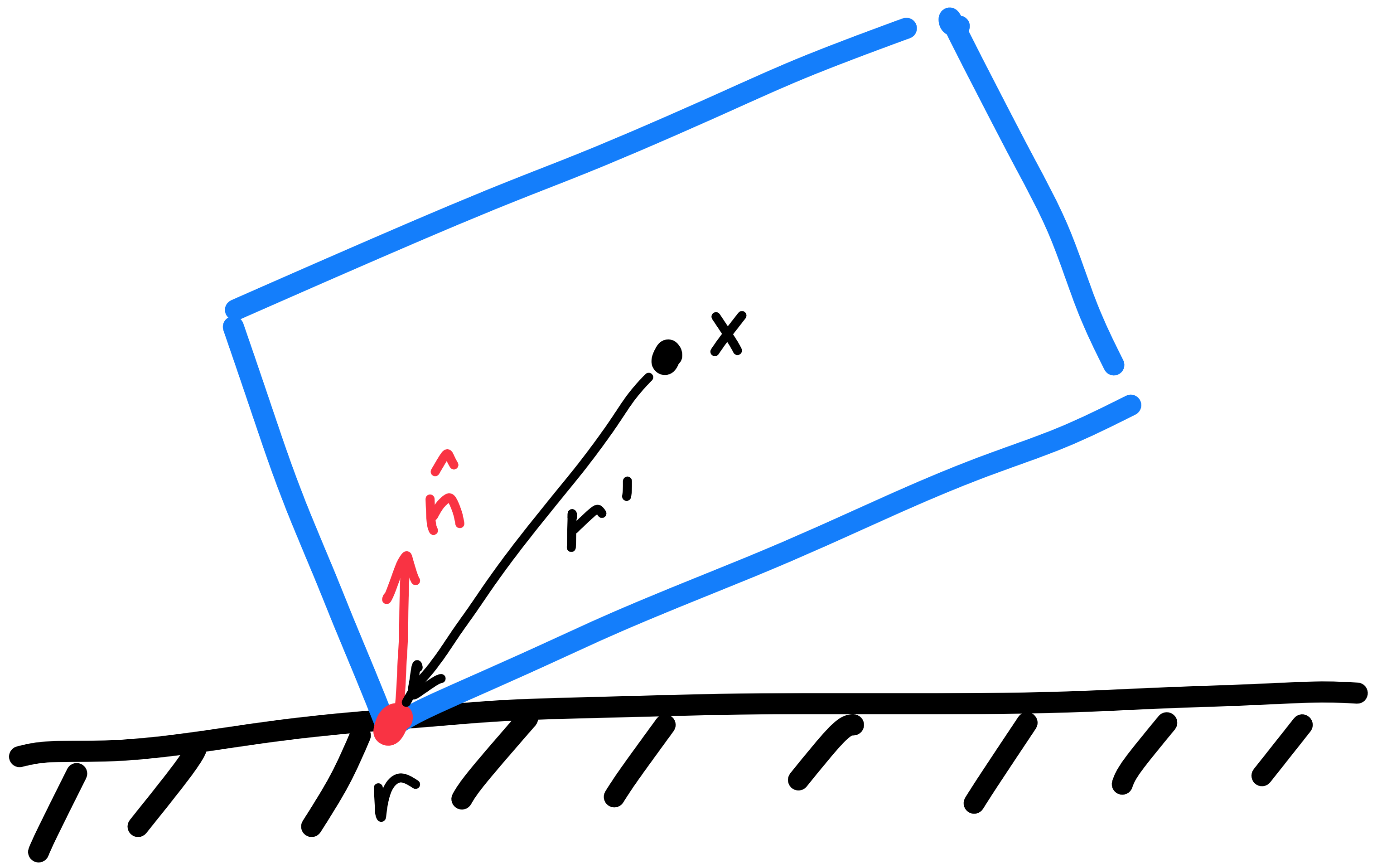
bar moves freely (no pivot, friction, etc.)

Collisions: rigid body–obstacle

Body collides with fixed obstacle

- want to apply an impulse at the point of contact so that $v_n^+ = -c_r v_n^-$
- before collision: $v_n^- = \hat{\mathbf{n}} \cdot (\mathbf{v}^- + \omega^- \times \mathbf{r}')$ where $\mathbf{r}' = \mathbf{r} - \mathbf{x}$
- impulse is along normal: $\mathbf{j} = \gamma \hat{\mathbf{n}}$
- after collision: $\mathbf{v}^+ = \mathbf{v}^- + m^{-1}\mathbf{j}$; $\omega^+ = \omega^- + I^{-1}\,\mathbf{r}' \times \mathbf{j}$
- relate normal velocities before and after to find γ:

$$v_n^+ = \hat{\mathbf{n}} \cdot \left(\mathbf{v}^- + m^{-1}\mathbf{j} + (\omega^- + I^{-1}\,\mathbf{r}' \times \mathbf{j}) \times \mathbf{r}'\right)$$
$$= \hat{\mathbf{n}} \cdot \left(\mathbf{v}^- + m^{-1}\gamma\hat{\mathbf{n}} + \omega^- \times \mathbf{r}' + I^{-1}\gamma(\mathbf{r}' \times \hat{\mathbf{n}}) \times \mathbf{r}'\right)$$
$$= v_n^- + \gamma\left(m^{-1} + \hat{\mathbf{n}} \cdot I^{-1}(\mathbf{r}' \times \hat{\mathbf{n}}) \times \mathbf{r}'\right)$$

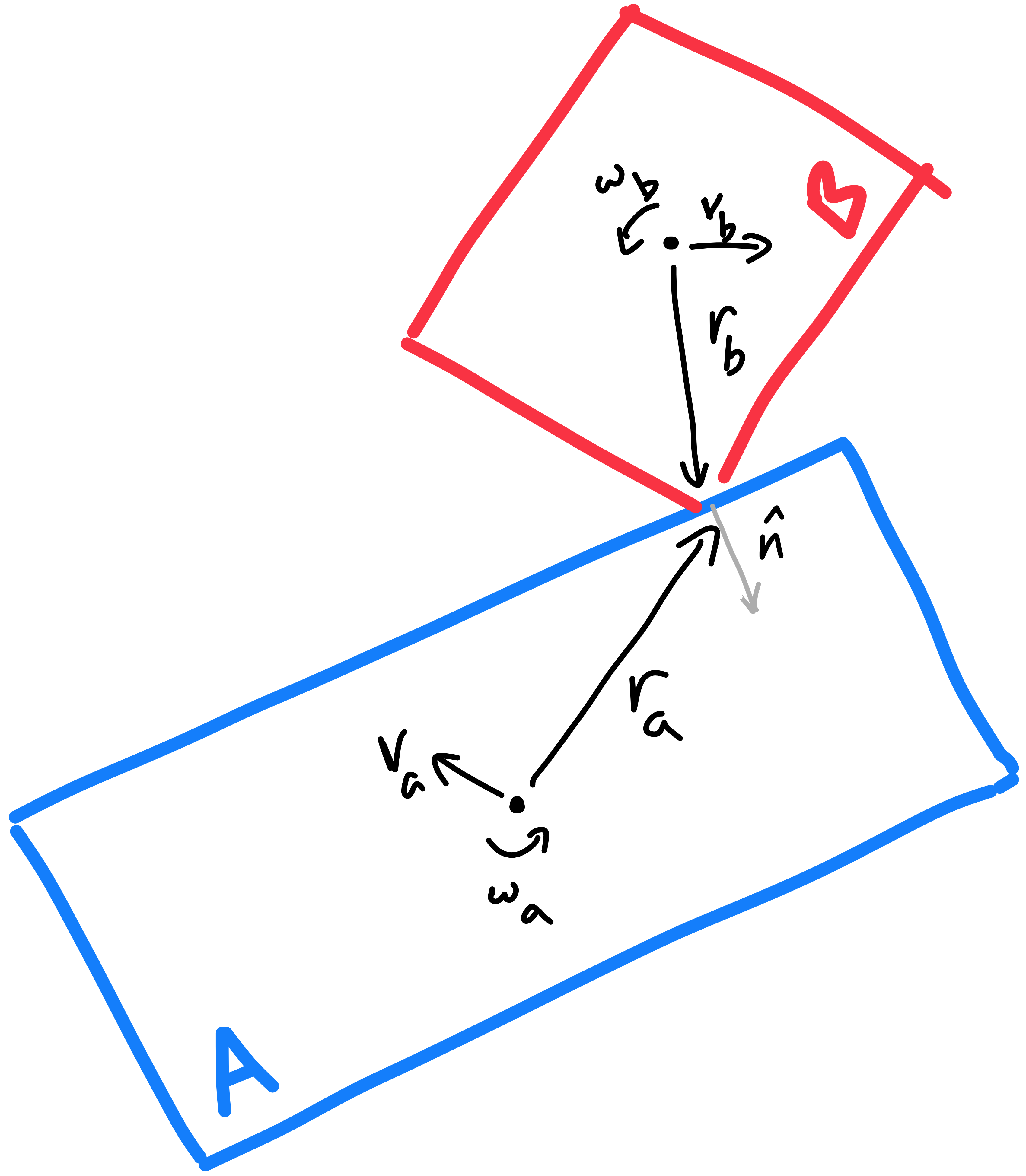


- so $\gamma = -(1 + c_r) m_{\text{eff}} v_n^-$ where $m_{\text{eff}} = \left(m^{-1} + \hat{\mathbf{n}} \cdot I^{-1}(\mathbf{r}' \times \hat{\mathbf{n}}) \times \mathbf{r}'\right)^{-1}$

Collisions: two rigid bodies

Bodies A and B collide at point $\mathbf{r}$

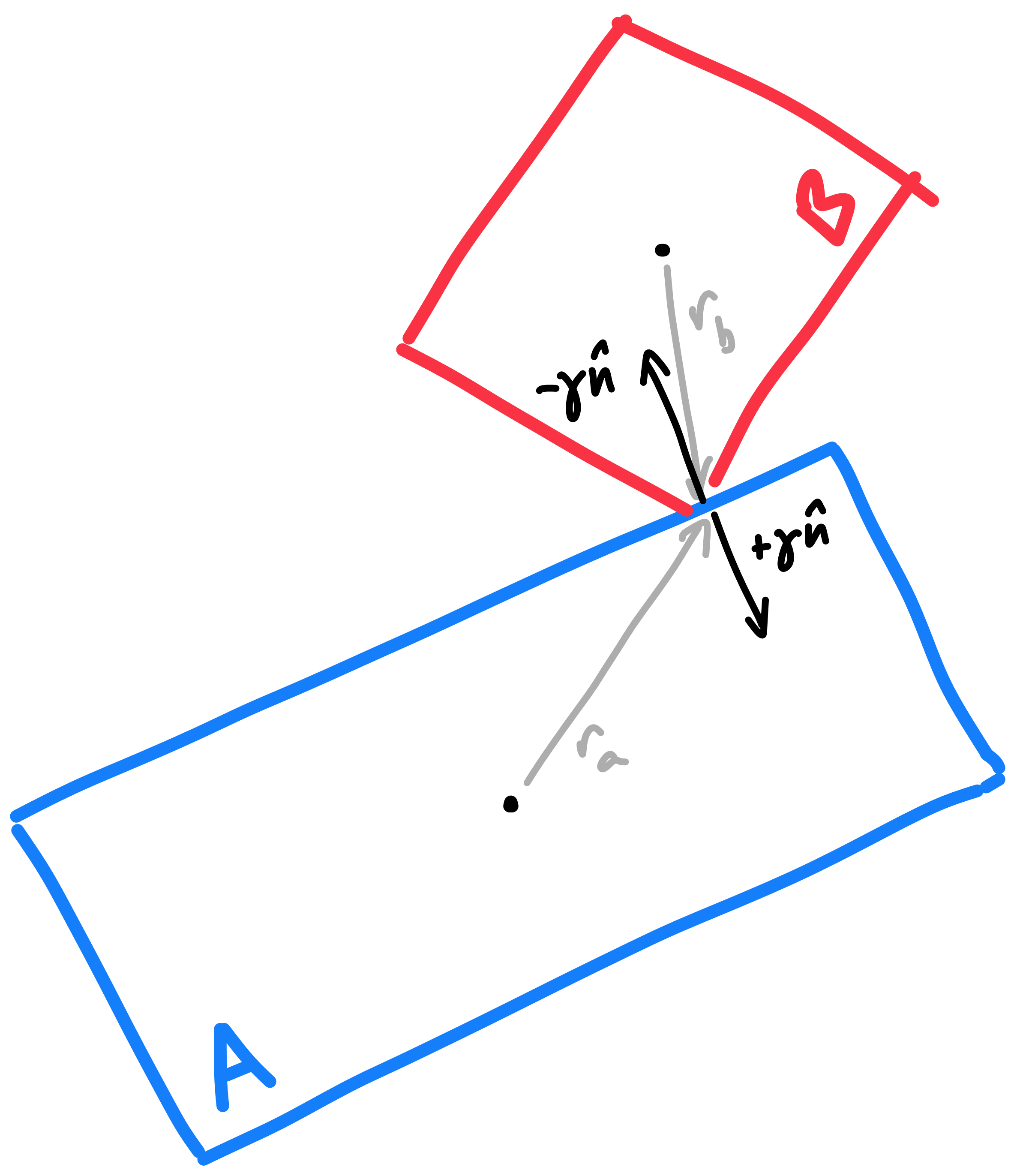
- pre-collision velocities are $\mathbf{v}_a, \omega_a, \mathbf{v}_b, \omega_b$
- velocity of colliding point on body A: $\mathbf{v}_a + \omega_a \times \mathbf{r}_a$
  where $\mathbf{r}_a = \mathbf{r} - \mathbf{x}_a$
- velocity of colliding point on body B: $\mathbf{v}_b + \omega_b \times \mathbf{r}_b$
  where $\mathbf{r}_b = \mathbf{r} - \mathbf{x}_b$
- relative normal velocity:
  $v_n = \hat{\mathbf{n}} \cdot \left(\mathbf{v}_a - \mathbf{v}_b + \omega_a \times \mathbf{r}_a - \omega_b \times \mathbf{r}_b\right)$
- will apply an impulse in the normal direction at the point of contact
- decide size of impulse using restitution hypothesis:
  $v_n^+ = -c_r v_n^-$



- will apply impulse $\mathbf{j}$ to body A and $-\mathbf{j}$ to body B, both at point $\mathbf{r}$
- for body A, $\Delta \mathbf{v}_a = m_a^{-1}\mathbf{j}$ and $\Delta\omega_a = I_a^{-1}\mathbf{r}_a \times \mathbf{j}$
- for body B, $\Delta \mathbf{v}_b = -\, m_b^{-1}\mathbf{j}$ and $\Delta\omega_b = -\, I_b^{-1}\mathbf{r}_b \times \mathbf{j}$
- the impulse is in the direction of the collision normal: $\mathbf{j} = \gamma\hat{\mathbf{n}}$
- so the post-collision relative velocity is

$$
\begin{aligned}
v_n^+ &= \hat{\mathbf{n}} \cdot \left(\mathbf{v}_a^+ - \mathbf{v}_b^+ + \omega_a^+ \times \mathbf{r}_a - \omega_b^+ \times \mathbf{r}_b\right) \\
&= v_n^- + \hat{\mathbf{n}} \cdot \left(\Delta\mathbf{v}_a - \Delta\mathbf{v}_b + \Delta\omega_a \times \mathbf{r}_a - \Delta\omega_b \times \mathbf{r}_b\right) \\
&= v_n^- + \hat{\mathbf{n}} \cdot \left(m_a^{-1}\gamma\hat{\mathbf{n}} + m_b^{-1}\gamma\hat{\mathbf{n}} + I_a^{-1}(\mathbf{r}_a \times \gamma\hat{\mathbf{n}}) \times \mathbf{r}_a + I_b^{-1}(\mathbf{r}_b \times \gamma\hat{\mathbf{n}}) \times \mathbf{r}_b\right) \\
&= v_n^- + \underbrace{\left(m_a^{-1} + m_b^{-1} + I_a^{-1}\hat{\mathbf{n}} \cdot (\mathbf{r}_a \times \hat{\mathbf{n}}) \times \mathbf{r}_a + I_b^{-1}\hat{\mathbf{n}} \cdot (\mathbf{r}_b \times \hat{\mathbf{n}}) \times \mathbf{r}_b\right)}_{m_{\text{eff}}^{-1}} \gamma
\end{aligned}
$$

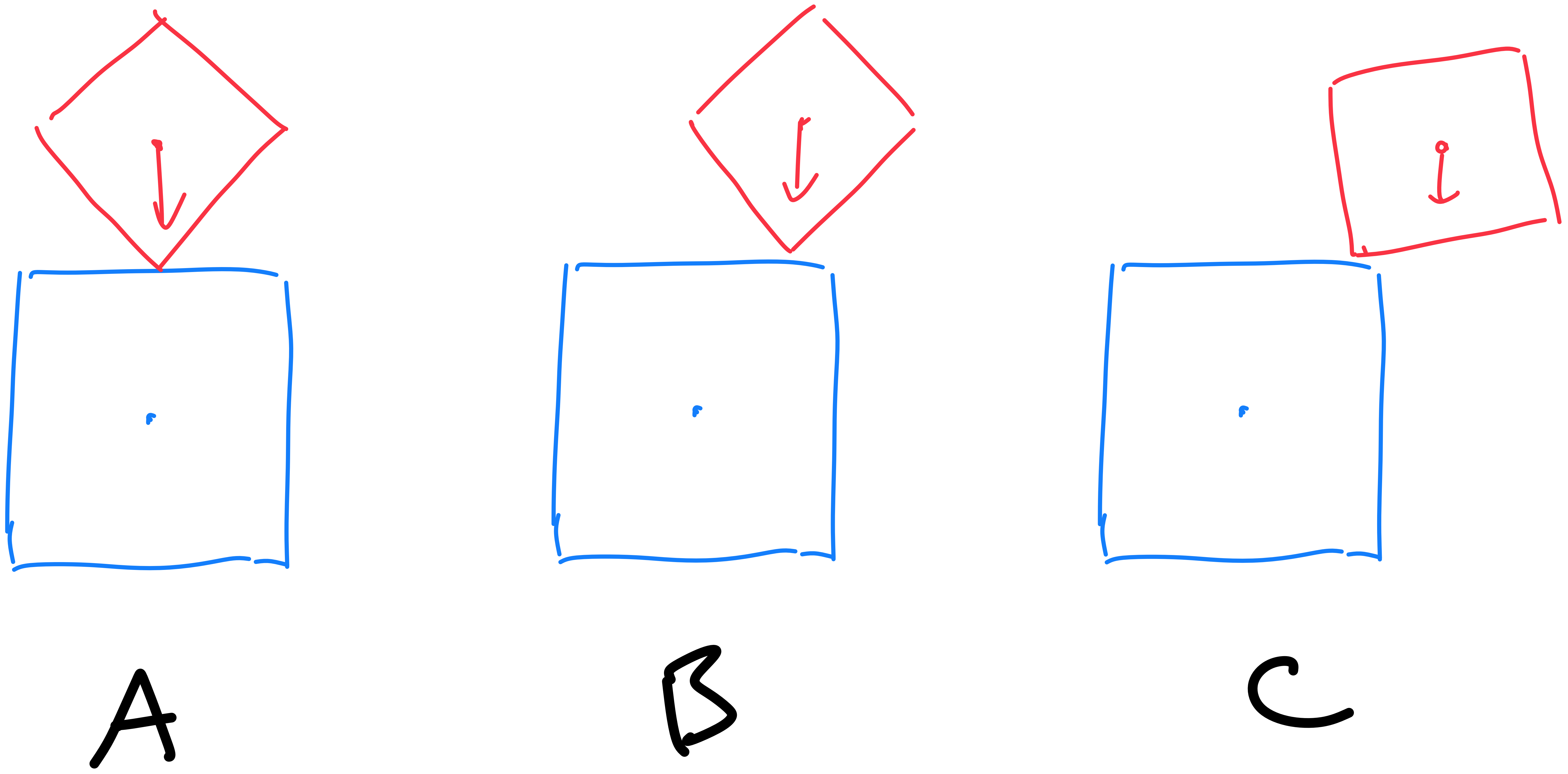
- setting $v_n^+ = -\, c_r v_n^-$ leads to $\gamma = -\,(1 + c_r)m_{\text{eff}}v_n^-$



poll: Three collisions

Same two bodies in all three cases, equal m and I, $c_r = 1$

- initial velocities are the same for all three; contact point is the same for B and C



Collision detection (overlap) for polygons

The easy case for overlap testing is convex polygons

- for convex shapes, a separating axis exists if and only if the polygons don’t overlap
- for convex polygons, if a separating axis exists then one of the edge normals is a separating axis
- so, to test two convex polygons for overlap:

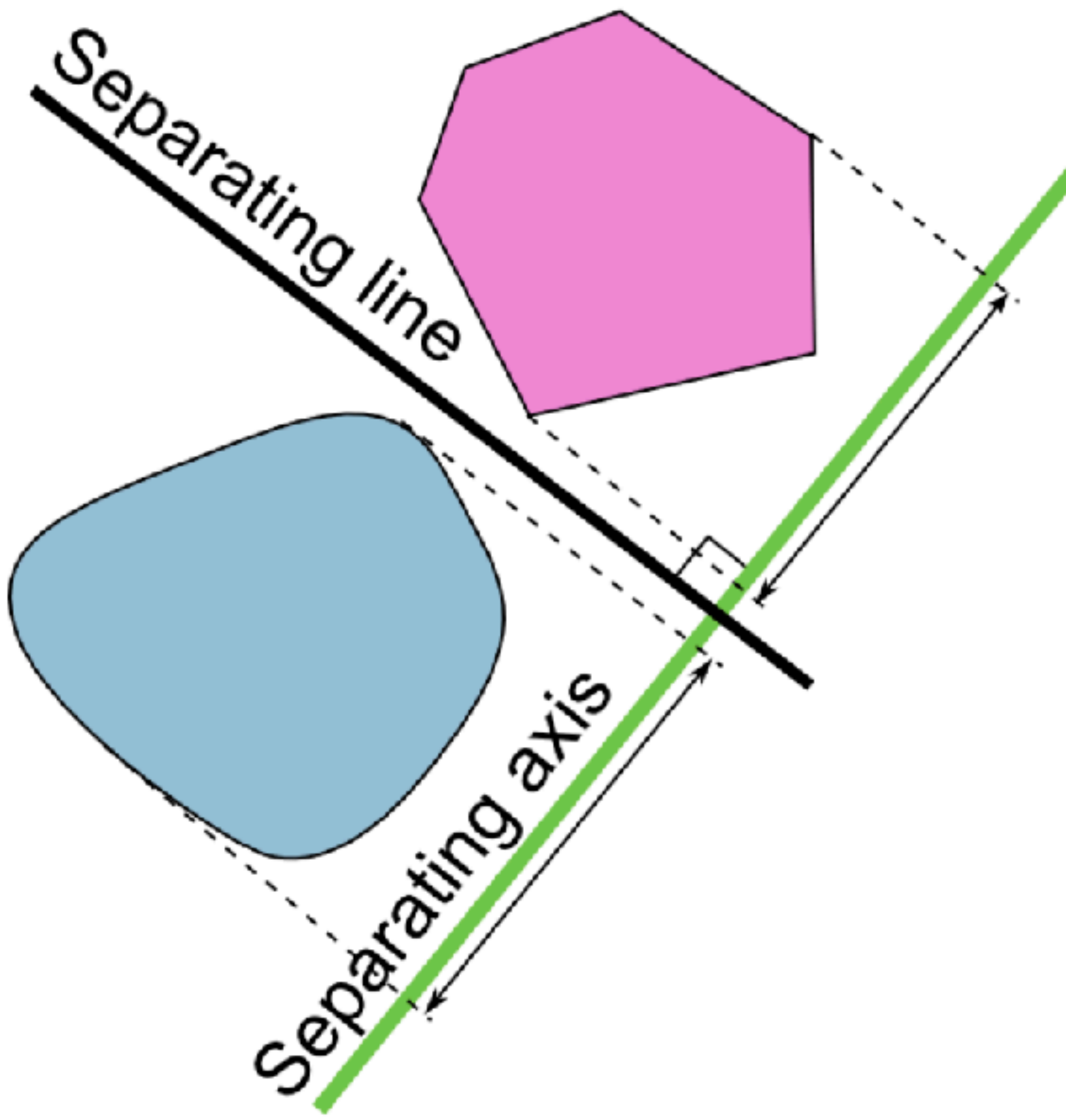
distance(e → x): normal(e) · (x – point_on(e))

separation(e → P): min of distance(e → v) for v in vertices(P)

separation(P → Q): max of separation(e → Q) for e in edges(P)

separation(P, Q): max(separation(P → Q), separation(Q → P))

overlap(P, Q): separation(P, Q) > 0



https://en.wikipedia.org/wiki/Hyperplane_separation_theorem

Collision detection (overlap) for polygons

- so, to test two convex polygons for overlap:

  distance(e → x): normal(e) · (x – point_on(e))

  separation(e → P): min of distance(e → v) for v in vertices(P)

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  separation(P, Q): max(separation(P → Q), separation(Q → P))

  overlap(P, Q): separation(P, Q) > 0

- …and for later use in collision computations, remember which vertex and edge produced the maximum minimum distance
  - we call this the “incident vertex” and the “reference edge”