CS 5220

Graph Algorithms

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Logistics

Remaining meetings

- Nov 26: Async iterations / final outline
- Dec 3 and 5: Project presentations

Course eval

- Open Dec 2-12
- Completion counts toward participation

Final deadline

Midnight on 12/19 (must be after 4:30 PM per university)

Final outline

- Think “outline of my paper”
- Should include
  - Basics of what you want to do
  - Evaluation setup (workload, etc)
  - Planned (or finished) performance experiments
  - Possibly a timeline
- Can represent reconfigured group

Project presentations

- Logistics
  - Think 3-5 minutes
  - Not everyone needs to talk!
  - Slides via Zoom share
- Contents
  - What you want to do / have done
  - Evaluation and performance plans
    - Does not have to be finished!

Notes

- Care most about *final report*
  - I don't want to stare at your code!
- Outline/presentation help me give feedback
- Emphasis is *performance analysis and tuning*
  - Think about benchmarks and baselines
  - Think about strong/weak scaling experiments
  - ... and check for correctness as well
- Get something done, fine to speculate on next steps

Lecture

Plan

- Some background on graphs
- Applications and building blocks
- Basic parallel graph algorithms
- Representations and performance
- Graphs and LA
- Frameworks

Graphs

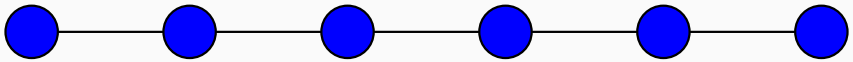
Mathematically: $G = (V, E)$ where $E \subset V \times V$

- Convention: $|V| = n$ and $|E| = m$
- May be directed or undirected
- May have weights $w_V : V \to \mathbb{R}$ or $w_E : E :\to \mathbb{R}$
- May have other node or edge attributes as well
- Path is $\left[(u_i, u_{i+1}) \right]_{i=1}^{\ell} \in E^*$, sum of weights is length
- Diameter is $\max_{s,t \in V} d(s, t)$

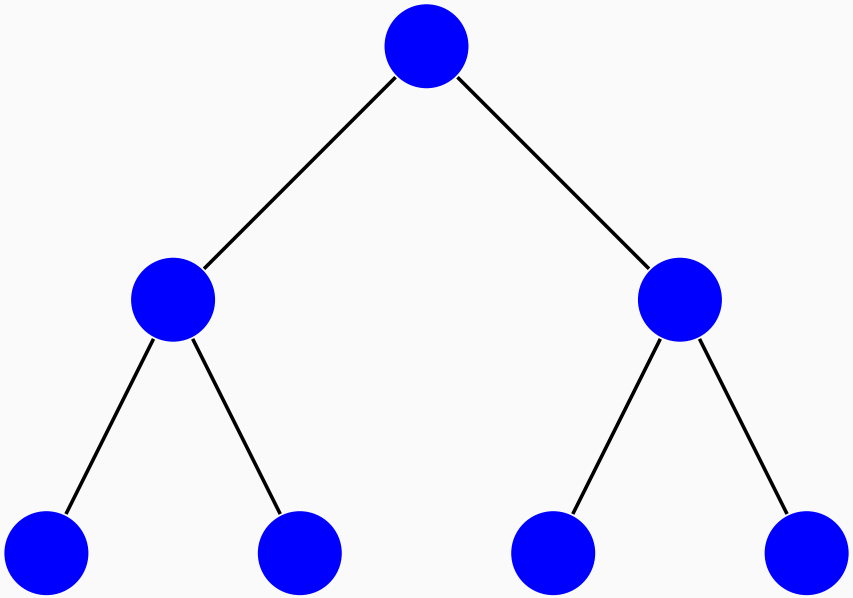
Generalizations

- Hypergraph (edges in V^d)
- Multigraph (multiple copies of edges)

Types of graphs



Types of graphs



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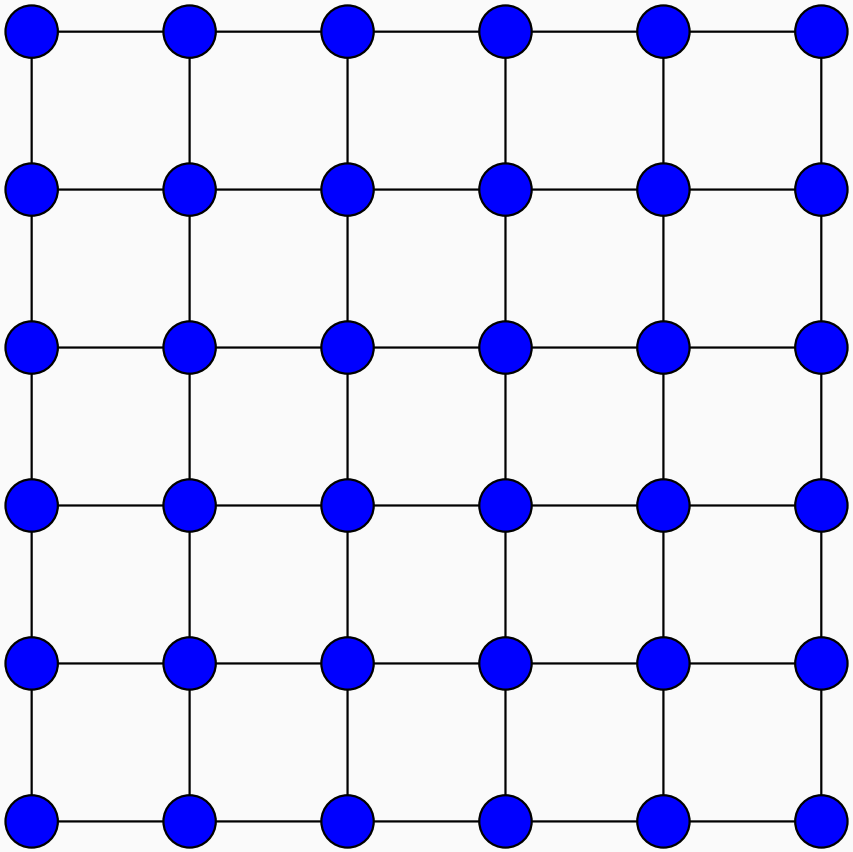

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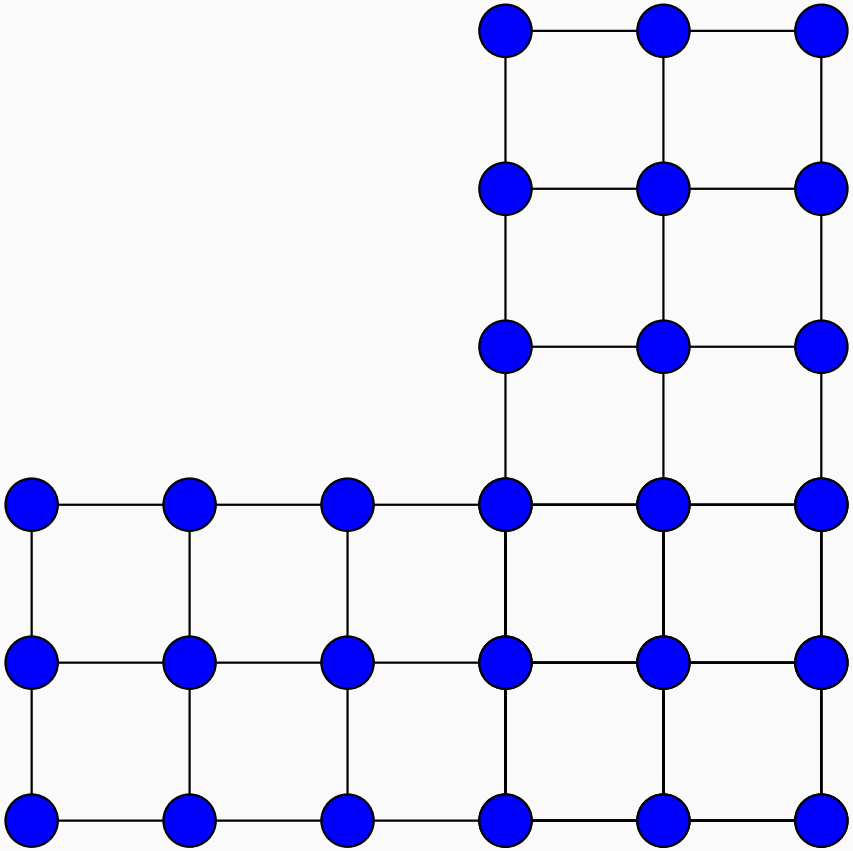

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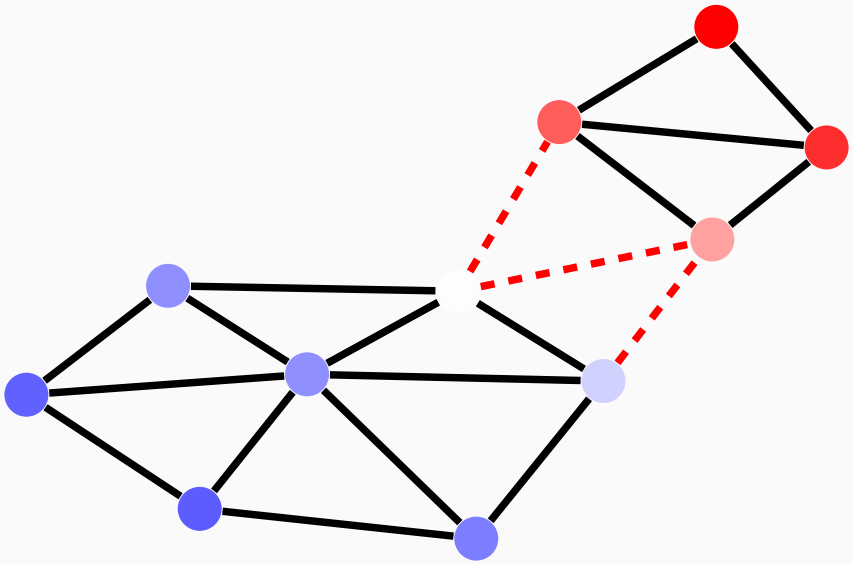

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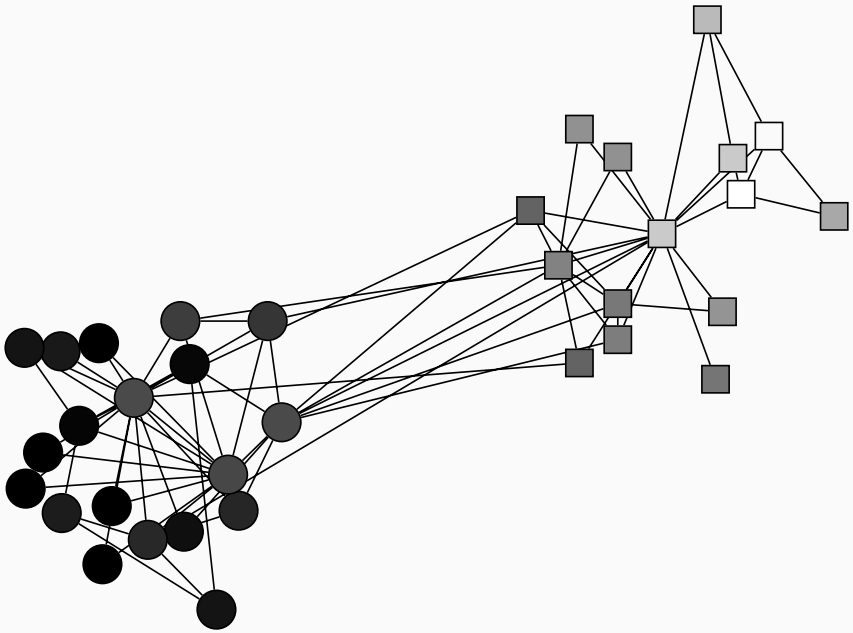


Types of graphs

Many possible structures:

- Lines and trees
- Completely regular grids
- Planar graphs (no edges need cross)
- Low-dimensional Euclidean
- Power law graphs
- ...

Algorithms are not one-size-fits-all!

Ends of a spectrum

	Planar	Power law
Vertex degree	Uniformly small	$P(\deg = k) \sim k^{-\gamma}$
Radius	$\Omega(\sqrt{n})$	Small
Edge sep	$O(\sqrt{n})$	nothing small
Linear solve	Direct OK	Iterative
Apps	PDEs	Social networks

Calls for different methods!

Applications: Routing and shortest paths

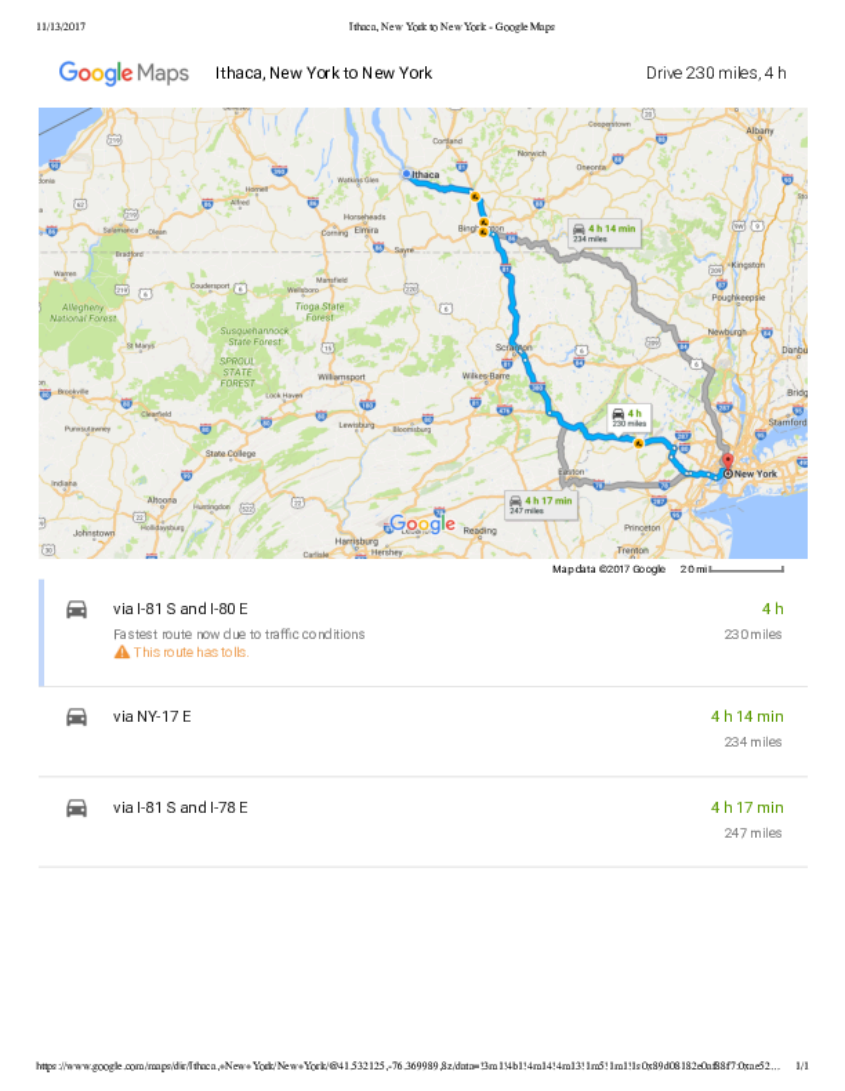


Figure 1: image

Applications: Traversal, ranking, clustering

- Web crawl / traversal
- PageRank, HITS
- Clustering similar documents

Applications: Sparse solvers

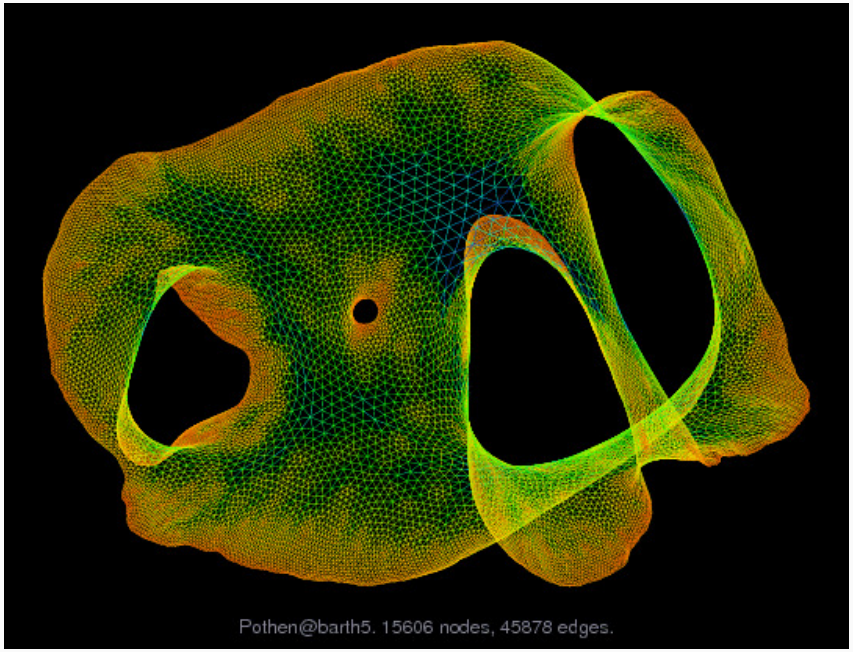


Figure 2: image

- Ordering for sparse factorization
- Partitioning
- Coarsening for AMG

Applications: Dimensionality reduction

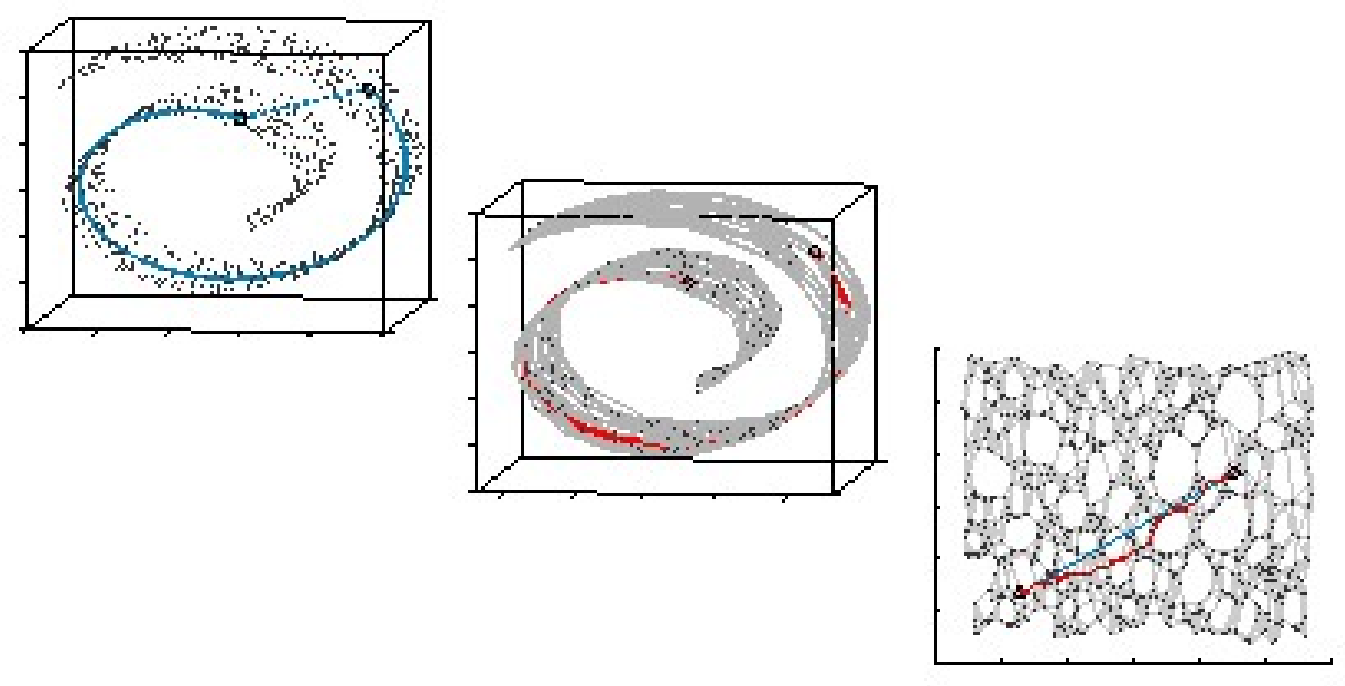


Figure 3: image

Common building blocks

- Traversals
- Shortest paths
- Spanning tree
- Flow computations
- Topological sort
- Coloring
- ...

... and most of sparse linear algebra.

Over-simple models

Let t_p = idealized time on p processors

- t_1 = work
- t_∞ = span (or depth, or critical path length)

Don't bother with parallel DFS! Span is $\Omega(n)$.

Let's spend a few minutes on more productive algorithms...

Serial BFS

- Push seed node onto queue and mark
- While Q nonempty
    - Pop node from queue
    - Visit node
    - Push unmarked neighbors on queue
    - Mark all neighbors

Parallel BFS

Simple idea: parallelize across frontiers

- Pro: Simple to think about
- Pro: Lots of parallelism with small radius?
- Con: What if frontiers are small?

Parallel BFS: Ullman-Yannakakis

Assuming a high-diameter graph:

- Form set S with start + random nodes, $|S| = \Theta(\sqrt{n} \log n)$
  - long shortest paths go through S w.h.p.
- Take $\sqrt{n}$ steps of BFS from each seed in S
- Form aux graph for distances between seeds
- Run all-pairs shortest path on aux graph

OK, but what if diameter is not large?

Serial BFS: Bottom-up

- Set $d[v] = \infty$ for all vertices
- Set $d[s] = 0$ for seed s
- Until d stops changing
    - For each $u \in V$
        - $d[u] = \min(d[u], \min_{w \in N(u)} d[w] + 1)$

Parallel BFS

Key ideas:

- At some point, switch from top-down expanding frontier (“are you my child?”) to bottom-up checking for parents (“are you my parent?”)
- Use 2D blocking of adjacency

Single-source shortest path

Classic algorithm: Dijkstra

- Dequeue closest point to frontier, expand frontier
- Update priority queue of distances (in parallel)
- Repeat

Or run serial Dijkstra from different sources for APSP.

Alternate idea: label correcting

Initialize $d[u]$ with distance over-estimates to source

- $d[s] = 0$
- Repeatedly relax $d[u] := \min_{(v,u) \in E} d[v] + w(v, u)$

Converges (eventually) as long as all nodes visited repeatedly, updates are atomic. If serial sweep in a consistent order, call it Bellman-Ford.

Single-source shortest path: Δ-stepping

Alternate approach: *hybrid* algorithm

- Process a “bucket” at a time
- Relax “light” edges (wt < Δ), might add to bucket
- When bucket empties, relax “heavy” edges a la Dijkstra

Maximal independent sets (MIS)

- $S \subset V$ *independent* if none are neighbors.
- *Maximal* if no others can be added and remain independent.
- *Maximum* if no other MIS is bigger.
- Maximum is NP-hard; maximal is easy (serial)

Simple greedy MIS

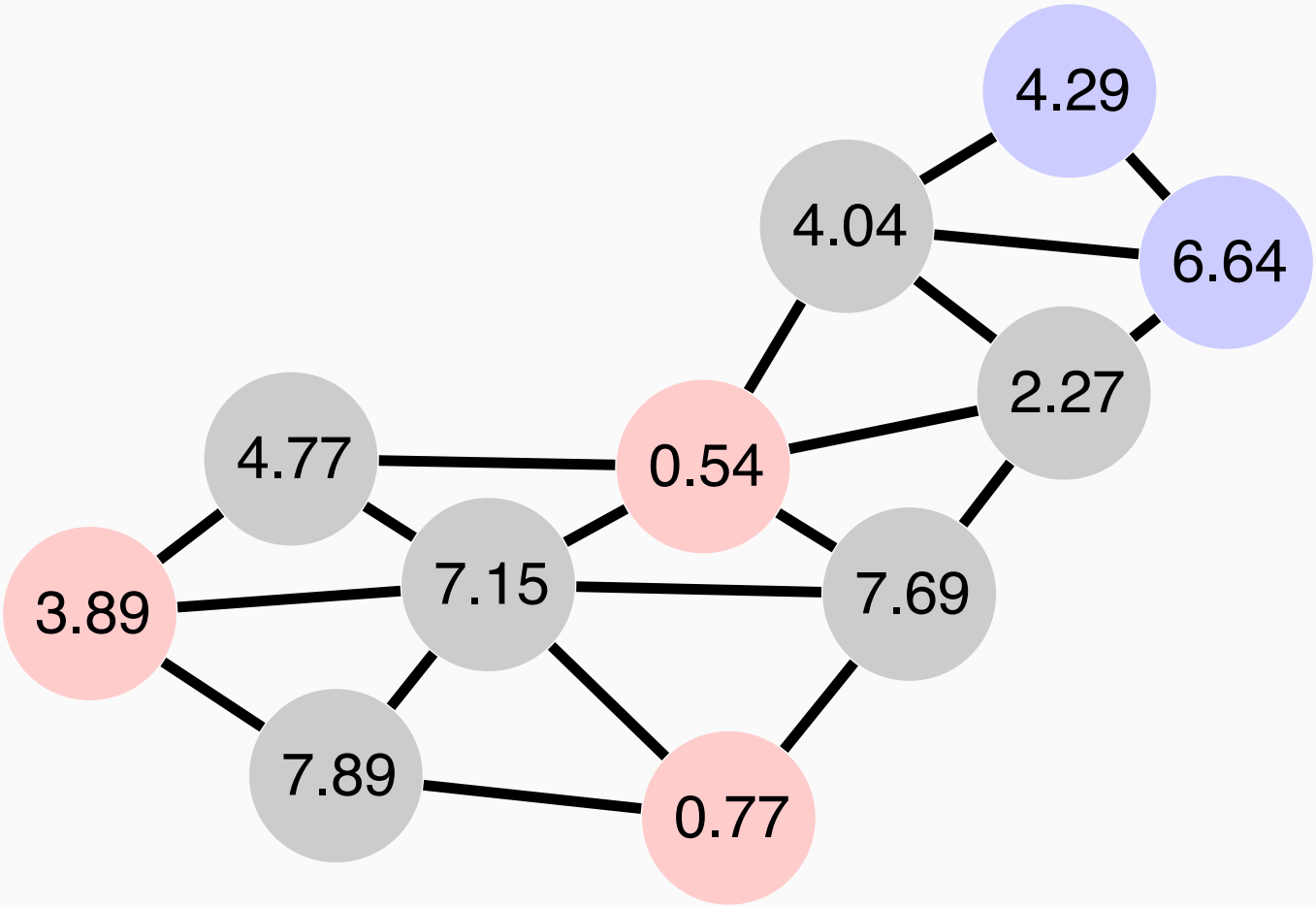
- Start with S empty
- For each $v \in V$ *sequentially*, add v to S if possible.

Luby's algorithm

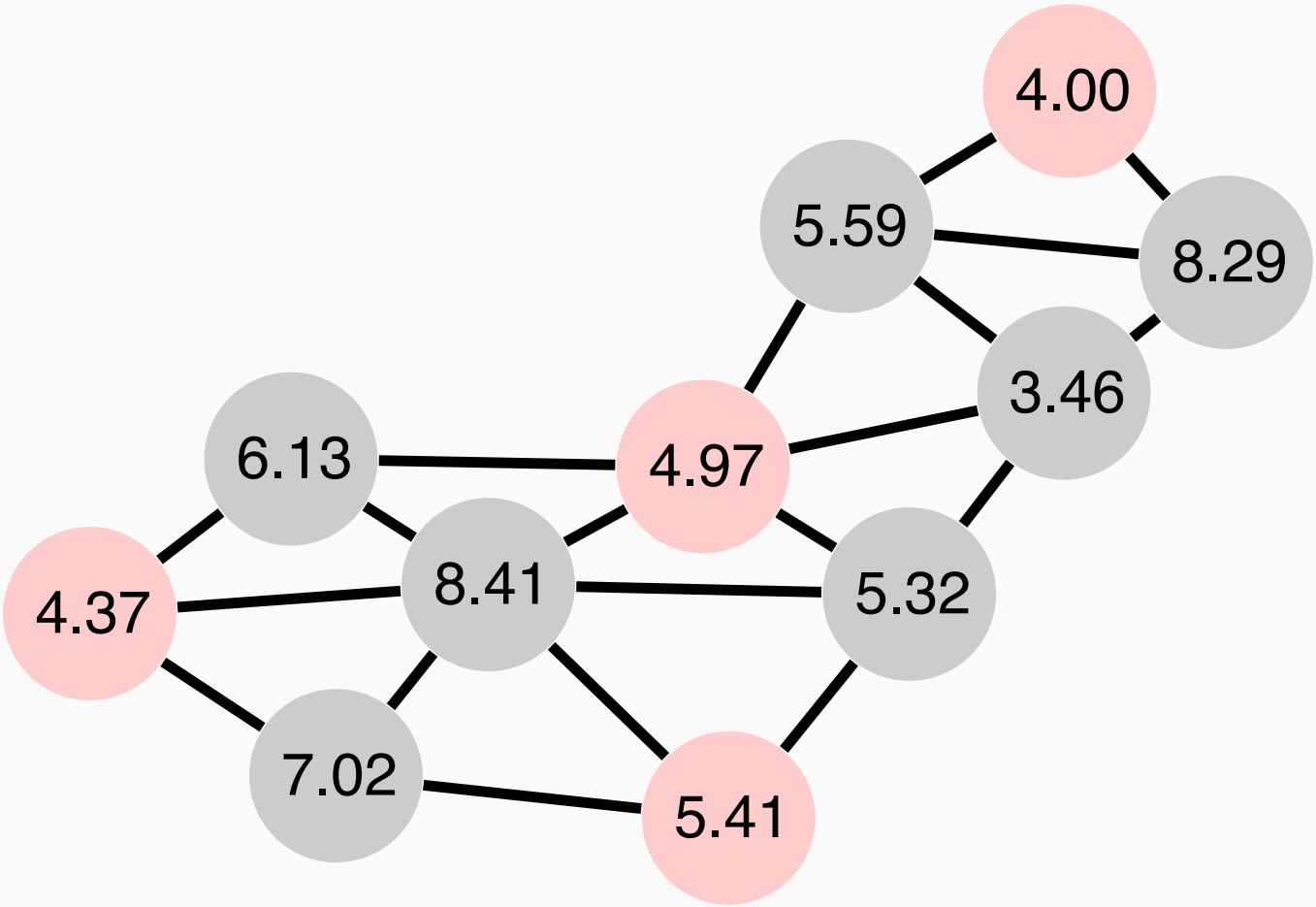
- Init $S := \emptyset$
- Init candidates $C := V$
- While $C \neq \emptyset$
  - Label each v with a random $r(v)$
  - For each $v \in C$ in parallel, if $r(v) < \min_{\mathcal{N}(v)} r(u)$
    - Move v from C to S
    - Remove neighbors from v to C

Very probably finishes in $O(\log n)$ rounds.

Luby's algorithm (round 1)



Luby's algorithm (round 2)



A fundamental problem

Many graph ops are

- Computationally cheap (per node or edge)
- Bad for locality

Memory bandwidth as a limiting factor.

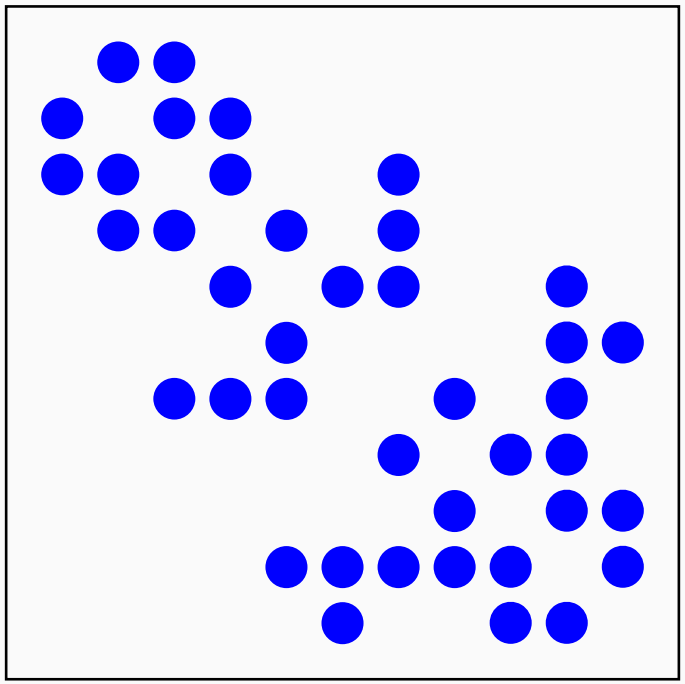
Big data?

Consider:

- 323 million in US (fits in 32-bit int)
- About 350 Facebook friends each
- Compressed sparse row: about 450 GB

Topology (no metadata) on one big cloud node...

Graph rep: Adj matrix



Pro: efficient for dense graphs

Con: wasteful for sparse case...

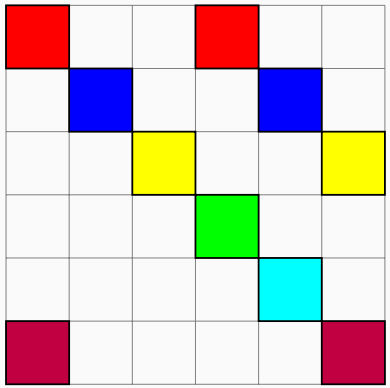
Graph rep: Coordinate

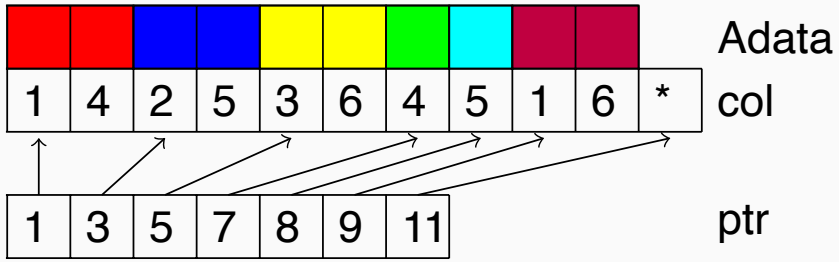
- Tuples: (i, j, w_{ij})
- Pro: Easy to update
- Con: Slow for multiply

Graph rep: Adj list

- Linked lists of adjacent nodes
- Pro: Still easy to update
- Con: May cost more to store than coord?

Graph rep: CSR





Pro: traversal? Con: updates

Graph rep: implicit

- Idea: Never materialize a graph data structure
- Key: Provide traversal primitives
- Pro: Explicit rep'n sometimes overkill for one-off graphs?
- Con: Hard to use canned software (except NLA?)

Graph algorithms and LA

- Really is standard LA
  - Spectral partitioning and clustering
  - PageRank and some other centralities
  - “Laplacian Paradigm” (Spielman, Teng, others...)
- Looks like LA
  - Floyd-Warshall
  - Breadth-first search?

Semirings have $\oplus$ and $\otimes$ s.t.

- Addition is commutative+associative with a 0
- Multiplication is associative with identity 1
- Both are distributive
- $a \otimes 0 = 0 \otimes a = 0$
- But no subtraction or division

Technically *modules* over semirings

Example: min-plus

- $\oplus = \min$ and additive identity $0 \equiv \infty$
- $\otimes = +$ and multiplicative identity $1 \equiv 0$
- Useful for shortest distance: $d = A \otimes d$

Graph BLAS

http://www.graphblas.org/

- Version 2.1.0 (final) as of Dec 2023
- (Opaque) internal sparse matrix data structure
- Allows operations over misc semirings

Graph frameworks

Several to choose from!

- Pregel, Apache Giraph, Stanford GPS, ...
- GraphLab family
  - GraphLab: Original distributed memory
  - PowerGraph: For “natural” (power law) networks
  - GraphChi: *Chi*huahua – shared mem vs distributed
- Outperformed by Galois, Ligra, BlockGRACE, others
- But... programming model was easy
- GraphIt - best of both worlds?

Graph frameworks

- "Think as a vertex"
  - Each vertex updates locally
  - Exchanges messages with neighbors
  - Runtime actually schedules updates/messages
- Message sent at super-step S arrives at $S + 1$
- Looks like BSP

At what COST?

"Scalability! But at what COST?"

McSherry, Isard, Murray, HotOS 15

> *You can have a second computer once you've shown you know how to use the first one.*
>
> *– Paul Barham (quoted in intro)*

- Configuration that Outperforms a Single Thread
- Observation: many systems have unbounded COST!