

**Definition 1** The relation “ $\varphi$  is an immediate subformula of  $\psi$ ” is the smallest relation such that

- $\varphi$  is an immediate subformula of  $\neg\varphi$
- $\varphi_1$  and  $\varphi_2$  are immediate subformulas of  $\varphi_1 \wedge \varphi_2$
- $\varphi_1$  and  $\varphi_2$  are immediate subformulas of  $\varphi_1 \vee \varphi_2$
- $\varphi_1$  and  $\varphi_2$  are immediate subformulas of  $\varphi_1 \Rightarrow \varphi_2$ .

The relation “ $\varphi$  is a subformula of  $\psi$ ” is the smallest relation such that

- $\varphi$  is a subformula of  $\varphi$
- if  $\varphi$  is an immediate subformula of  $\psi$ , then  $\varphi$  is a subformula of  $\psi$
- if  $\varphi$  is a subformula of  $\psi$  and  $\psi$  is a subformula of  $\gamma$ , then  $\varphi$  is a subformula of  $\gamma$ .

The only formulas having no immediate subformulas are propositional variables (that is,  $\varphi$  is an immediate subformula of  $p$  never holds). Propositional variables are often called *atomic formulas*. Other formulas are often called *compound formulas*. We say a propositional variable  $p$  occurs in  $\varphi$  if  $p$  is a subformula of  $\varphi$ .

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**Definition 2** The degree of a formula is defined by the following (primitive) recursive function:

$degree(\varphi) = \mathbf{case} \varphi \mathbf{ of}$   
     $\langle var, p \rangle \longrightarrow 0$   
     $\langle not, \psi \rangle \longrightarrow degree\psi + 1$   
     $\langle and, \psi_1, \psi_2 \rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1$   
     $\langle or, \psi_1, \psi_2 \rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1$   
     $\langle imp, \psi_1, \psi_2 \rangle \longrightarrow degree\psi_1 + degree\psi_2 + 1$   
**end.**

For example,  $p \wedge (q \vee \neg r)$  has degree 3, while  $p \wedge (q \vee r)$  has degree 2.

**Proposition 3**  $\varphi$  is an atomic formula (i.e., a propositional variable) if and only if  $\text{degree}(\varphi) = 0$ .

*Proof.* Immediate from the definition of degree. □

The degree of a formula lets us prove facts about the set *Form* of all formulas by induction on the degree of formulas.

**Proposition 4** *The Induction Principle for Formulas* Let  $P$  be a property of formulas. If

(i)  $P(\varphi)$  holds for every formula of degree 0;

(ii) for all  $n > 0$ , if  $P(\varphi)$  holds for every formula  $\varphi$  of degree  $< n$ , then  $P(\varphi)$  holds for every formula of degree  $n$ ;

Then  $P(\varphi)$  holds for every formula  $\varphi$ .

*Proof.* Let  $X$  be the set  $\{\varphi \mid P(\varphi) \text{ does not hold}\}$ . We want to show that  $P$  holds for all formulas  $\varphi$ , i.e., that  $X$  is empty.

We proceed by contradiction. Assume  $X$  not empty. By a well-known property of the natural numbers, there exists a formula  $\varphi_0 \in X$  that has minimal degree  $n_0$ , i.e., such that there is no formula in  $X$  with a smaller degree (there could be other formulas with the same degree). Let  $\varphi$  be an arbitrary formula  $\varphi$  with degree  $< n_0$ . Since  $\text{degree}(\varphi) < \text{degree}(\varphi_0)$ ,  $\varphi$  cannot be in  $X$ , therefore  $P(\varphi)$  holds. Since  $\varphi$  was arbitrary,  $P(\varphi)$  holds for all  $\varphi$ s with degree less than  $n_0$ . By property (ii), then, this means that  $P(\varphi_0)$  holds, i.e.,  $\varphi_0 \notin X$ , a contradiction. Therefore,  $X$  is empty, as required. □

**Proposition 5** For every formula  $\varphi$ , the set  $\text{Sub}(\varphi) = \{\psi \mid \psi \text{ is a subformula of } \varphi\}$  is finite.

*Proof.* By using the Principle of Induction for Formulas.

First, we check the base case. If  $\varphi$  has degree 0, then  $\varphi$  is a propositional variable, and  $\text{Sub}(\varphi) = \{\varphi\}$ , which is finite.

Second, let  $n > 0$ , and assume for all formulas  $\varphi$  of degree  $< n$ , that  $\text{Sub}(\varphi)$  is finite. Let  $\varphi$  be a formula of degree  $n$ . Since  $n > 0$ ,  $\varphi$  is a compound formula, and thus either of the form  $\neg\psi$ ,  $\psi_1 \wedge \psi_2$ ,  $\psi_1 \vee \psi_2$ , or  $\psi_1 \Rightarrow \psi_2$ . If  $\varphi$  is  $\neg\psi$ , then  $\text{degree}(\psi) = n - 1 < n$ , therefore by induction hypothesis,  $\text{Sub}(\psi)$  is finite. Since  $\text{Sub}(\varphi) = \text{Sub}(\psi) \cup \{\varphi\}$ ,  $\text{Sub}(\varphi)$  is finite. If  $\varphi$  is  $\psi_1 \wedge \psi_2$ , then  $\text{degree}(\psi_1)$  and  $\text{degree}(\psi_2)$  are both  $< n$ , and by the induction hypothesis, we have  $\text{Sub}(\psi_1)$  and

$Sub(\psi_2)$  finite; since  $Sub(\varphi) = Sub(\psi_1) \cup Sub(\psi_2) \cup \{\varphi\}$ ,  $Sub(\varphi)$  is finite. A similar argument works for  $\vee$  and  $\Rightarrow$ .  $\square$

Assume a set  $\mathbb{B} = \{t, f\}$  of *truth values*. Let  $S$  be a set of formulas.

**Definition 6** A *valuation*  $v$  on  $S$  is a function  $v : S \rightarrow \mathbb{B}$ .

We say  $\varphi$  is true under valuation  $v$  if  $v(\varphi) = t$ . Similarly,  $\varphi$  is false under valuation  $v$  if  $v(\varphi) = f$ .

**Definition 7** A *Boolean valuation*  $v$  is a valuation on  $Form$  such that:

- $v(\neg\varphi) = t$  if and only if  $v(\varphi) = f$
- $v(\varphi \wedge \psi) = t$  if and only if  $v(\varphi) = t$  and  $v(\psi) = t$
- $v(\varphi \vee \psi) = t$  if and only if  $v(\varphi) = t$  or  $v(\psi) = t$
- $v(\varphi \Rightarrow \psi) = t$  if and only if when  $v(\varphi) = t$ , then  $v(\psi) = t$ .

Given two valuations  $v_1, v_2$ , if  $v_1(\varphi) = v_2(\varphi)$ , then  $v_1$  and  $v_2$  *agree on*  $\varphi$ . If  $v_1$  and  $v_2$  agree on all formulas in a set  $S$ , then  $v_1$  and  $v_2$  *agree on*  $S$ .

Let  $S_1$  and  $S_2$  be sets of formulas with  $S_1 \subseteq S_2$ . If  $v_1$  is a valuation on  $S_1$ ,  $v_2$  is a valuation on  $S_2$ , and  $v_1$  and  $v_2$  agree on  $S_1$ , then  $v_2$  is an *extension* of  $v_1$ .

An *interpretation*  $v_0$  is a valuation on propositional variables.

**Proposition 8** Let  $v_0$  be an interpretation. If  $v$  and  $v'$  are Boolean valuations that extend  $v_0$ , then  $v$  and  $v'$  agree on all formulas.

*Proof.* By induction on formulas.  $\square$

Thus, an interpretation can extend to *at most* a single Boolean valuation.

We can construct such a valuation explicitly:

$$\begin{aligned}
 \text{value}(\varphi, v_0) = & \text{case } \varphi \text{ of} \\
 & \langle \text{var}, p \rangle \longrightarrow v_0(p) \\
 & \langle \text{not}, \psi \rangle \longrightarrow \text{vnot}(\text{value}(\psi, v_0)) \\
 & \langle \text{and}, \psi_1, \psi_2 \rangle \longrightarrow \text{vand}(\text{value}(\psi_1, v_0), \text{value}(\psi_2, v_0)) \\
 & \langle \text{or}, \psi_1, \psi_2 \rangle \longrightarrow \text{vor}(\text{value}(\psi_1, v_0), \text{value}(\psi_2, v_0)) \\
 & \langle \text{imp}, \psi_1, \psi_2 \rangle \longrightarrow \text{vimp}(\text{value}(\psi_1, v_0), \text{value}(\psi_2, v_0)) \\
 & \text{end.}
 \end{aligned}$$

where  $vnot(t) = f$ ,  $vnot(f) = t$ ;  $vand(t, t) = t$ ,  $vand(t, f) = vand(f, t) = vand(f, f) = f$ ;  $vor(t, t) = vor(t, f) = vor(f, t) = t$ ,  $vor(f, f) = f$ ; and  $vimp(t, t) = vimp(f, t) = vimp(f, f) = t$ ,  $vimp(t, f) = f$ .

**Proposition 9** For every interpretation  $v_0$ ,  $value(-, v_0)$  is a Boolean valuation that extends  $v_0$ .

*Proof.* By induction on formulas. □

**Proposition 10** For every interpretation  $v_0$ , there is a unique Boolean valuation that extends  $v_0$ , namely,  $value(-, v_0)$ .

*Proof.* Combining the previous two propositions. □

We often write  $v_0 \models \varphi$  for  $value(\varphi, v_0) = t$ , and say that  $\varphi$  is true under interpretation  $v_0$ . Similarly, we write  $v_0 \not\models \varphi$  for  $value(\varphi, v_0) = f$ , and say that  $\varphi$  is false under interpretation  $v_0$ .

**Proposition 11** Let  $\varphi$  be a formula. If the interpretations  $v_0$  and  $v'_0$  agree on all propositional variables that occur in  $\varphi$ , then  $value(\varphi, v_0) = value(\varphi, v'_0)$ .

*Proof.* A straightforward induction will not quite work. We need to prove the slightly stronger statement: if the interpretations  $v_0$  and  $v'_0$  agree on all propositional variables that occur in  $\varphi$ , then for all subformulas  $\psi$  of  $\varphi$ ,  $value(\psi, v_0) = value(\psi, v'_0)$ . Clearly, this implies the result we want, since  $\varphi$  is a subformula of  $\varphi$ . And establishing the stronger result is a simple application of induction on formulas. □