24 Mar The Probabilistic Method

Recall: $R(k,\ell) =$ minimum n s.t. every graph with n vertices has a clique on k vertices or an independent set on ℓ vertices.

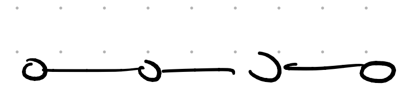
Last time: $R(k,\ell) \leq 2^{k+\ell-3}$

Today: $R(k,k) \geq 2^{(k+1)/2}$ (if $k \geq 3$)

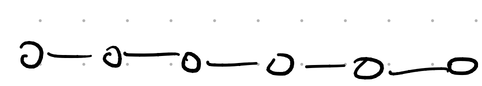
Def. A k-Ramsey graph is one with no clique of size k nor any indep set of size k.

Plan: Prove that for $n \geq 2^{\frac{k+1}{2}}$ a random sample from $G(n, \frac{1}{2})$ has pos prob of being a k-Ramsey graph.

k = 3: $2^{\frac{k+1}{2}} = 4$



k = 4: $2^{\frac{k+1}{2}} = 4\sqrt{2} \approx 5.6$



k ≥ 5: $\binom{n}{k} = \dfrac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} < \dfrac{1}{2}\cdot\left(\dfrac{n}{2}\right)^k$

because $k! > 2^{k+1}$.

Expected # of k-cliques in $G(n,\frac{1}{2})$.

$$\binom{n}{k} \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = \binom{n}{k} \cdot 2^{-\frac{1}{2} \cdot k \cdot (k-1)}$$

$$< \frac{1}{2} \cdot \left(\frac{n}{2}\right)^{k} \cdot \left[2^{-\frac{k-1}{2}}\right]^{k}$$

$$= \frac{1}{2} \cdot \left[\frac{n}{2 \cdot 2^{\frac{k-1}{2}}}\right]^{k}$$

$$\mathbb{E}[\text{\# k-cliques}] < \frac{1}{2} \cdot \left[\frac{n}{2^{\frac{k+1}{2}}}\right]^{k}$$

$$\mathbb{E}[\text{\# k ind sets}] < \frac{1}{2} \cdot \left[\frac{n}{2^{\frac{k+1}{2}}}\right]^{k}$$

$\Pr(\exists$ at least clique or ind set of size $k)$

$$< \left[\frac{n}{2^{\frac{k+1}{2}}}\right]^{k}$$

If $n < 2^{\frac{k+1}{2}}$ this probability is < 1.

$\Pr(G(n,\frac{1}{2})$ is a k-Ramsey graph$)$

$$> 1 - \left(\frac{n}{2^{\frac{k+1}{2}}}\right)^{k}.$$

Not only do k-Ramsey graphs exist,

when $n < 2^{k/2}$ they are INCREDIBLY PLENTIFUL! Almost every graph on n vertices is k-Ramsey.

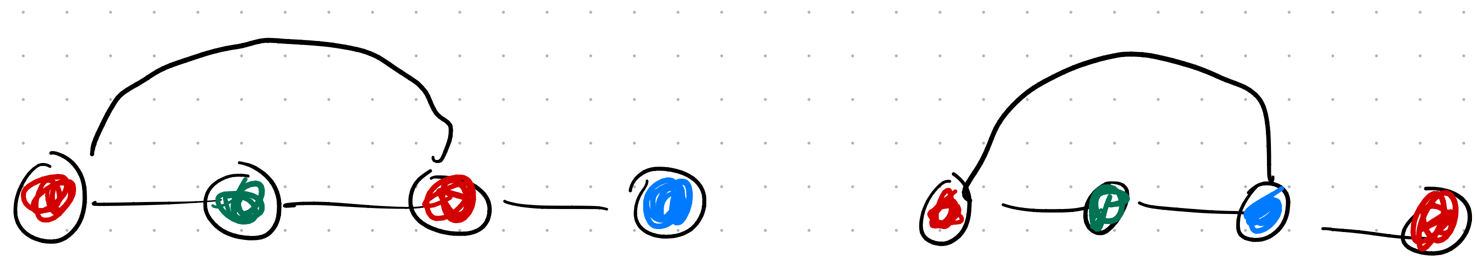
Explicit constructions. k-Ramsey graphs with 2^{k^c} (c a constant > 0) vertices. (Xin Li, 2023)

(Previously Chattopadhyay & Zuckerman)

Girth and Chromatic Number

Def. A k-coloring of a graph is a function from $V(G) \longrightarrow C$ where C is a set of k colors.

A k-coloring is proper if the endpoints of every edge have distinct colors.

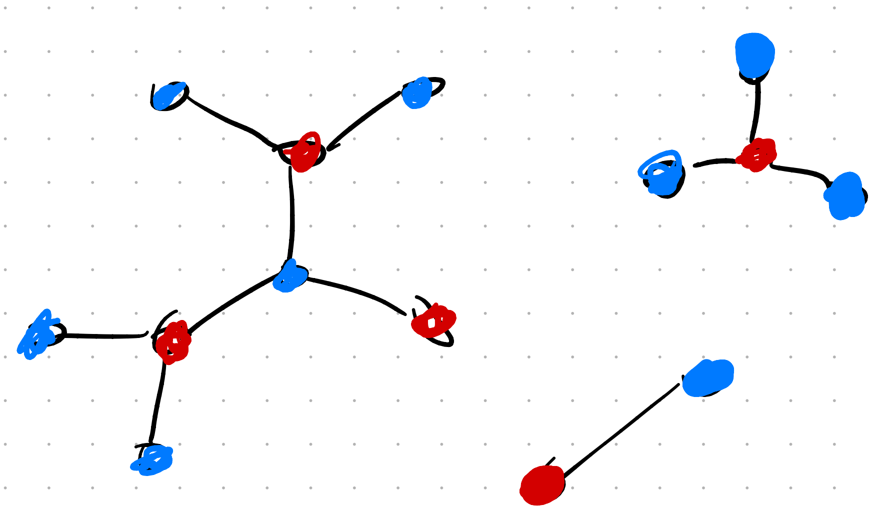


If G has clique of size $k+1$ it cannot have a proper k-coloring.
(Pigeonhole $\Rightarrow$ 2 vertices of the clique have same color)

<u>Def.</u> The <u>chromatic number</u> $\chi(G)$ is the minimum k s.t. G has a proper k-coloring.

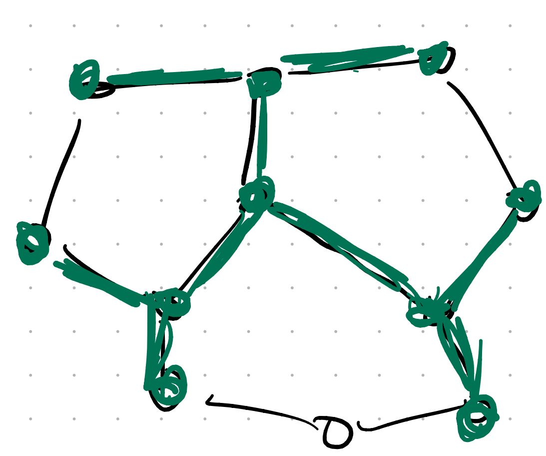
<u>Def.</u> The <u>girth</u> $\gamma(G)$ is the number of edges in the shortest cycle. Or $\gamma(G) = \infty$ if G has no cycles.

Example of $\gamma(G) = \infty$:



If $\gamma(G) = \infty$ then $\chi(G) = 2$.

Example of $\gamma(G) = 5$:



Graph of high girth are "locally tree-like."

If $\gamma(G) > 2r$ then a BFS around any vertex with "search radius r" will not discover a cycle.

<u>Theorem</u> (Erdős) For all $g, k < \infty$ there exist finite graphs G with $\gamma(G) > g$ and $\chi(G) > k$.

<u>Proof.</u> Choose p carefully and show $G(n,p)$ has a large subgraph with both of these properties with pos. probability.

Expected # of cycles of length $\leq g$.

Assume $\frac{1}{n} \leq p \leq \frac{1}{n}\left(\frac{n}{4g}\right)^{1/g}$.

Every cycle of length ℓ consists of

$v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v_0$

such that (v_{i-1}, v_i) is present in $G(n,p)$ for $i = 1, 2, \ldots, \ell$.

such sequences $\leq n^\ell$.

$\Pr(\text{all } \ell \text{ edges of the cycle are present}) = p^\ell$.

$\mathbb{E}[\#\text{ cycles of length } \ell] \leq n^\ell \cdot p^\ell$.

$$\mathbb{E}[\#\text{ cycles of length} \leq g] \leq \sum_{\ell=3}^{g} (pn)^\ell$$

$$\leq g \cdot (pn)^g$$

$pn \leq \left(\frac{n}{4g}\right)^{1/g}$

$(pn)^g \leq \frac{n}{4g}$

$$\leq \frac{n}{4}$$

MARKOV $\Longrightarrow$ with probability $\geq \frac{1}{2}$,

$G(n,p)$ has $\leq \frac{n}{2}$ cycles of length $\leq g$.

Thinking about chromatic number.... a proper k-coloring of G partitions its vertex set into k color classes (all the vertices labeled with a particular color) and each of them is an independent set in G.

Next step: We will choose p large enough that, probably, max indep set of G has $< \frac{n}{2k}$ vertices.

$$\mathbb{E}[\#\text{ indep sets of size } \tfrac{n}{2k}]$$

Assume n divisible by $2k$...

$t = \frac{n}{2k}$

$$= \binom{n}{n/2k} \cdot (1-p)^{\left(\frac{n}{2k}\right)\left(\frac{n}{2k}-1\right)/2}$$

$$< n^t \cdot e^{-pt(t-1)/2}$$

$$= \left[\frac{n}{e^{p(t-1)/2}}\right]^t .$$

Choose p to make $\frac{n}{e^{p(t-1)/2}} \leq \frac{1}{2}$

$$2n \leq e^{p(t-1)/2}$$

$$\ln(2n) \leq \frac{p}{2} \cdot (t-1) = \frac{p}{2} \cdot \left(\frac{n}{2k} - 1\right)$$

$$= \frac{p \cdot (n-2k)}{4k} .$$

$$p \geq \frac{4k \ln(2n)}{n-2k}$$

If n is lg enough that $n^{1/g} >> 4k\ln(2n)$ this inequalities are satisfiable!

Summarizing: If $\boxed{\frac{4k \ln(2n)}{n-2k} \leq p \leq \frac{1}{n} \cdot \left(\frac{n}{4g}\right)^{1/g}}$

then with probability $\geq \frac{1}{2}$, $G(n,p)$ has at most $\frac{n}{2}$ cycles of length $\leq g$.

Also with probability $> \frac{1}{2}$ $G(n,p)$ has no indep sets of size $\frac{n}{2k}$ or larger.

With prob > 0 both of these happen.

Then let $G_1 = G$ with one vertex deleted from each $\leq g$ cycle.

$|V(G_1)| \geq \frac{n}{2}$. G_1 has no indep sets of size $\geq \frac{n}{2k}$

$\rightarrow \chi(G_1) > k$.

G_1 has no cycle of length $\leq g$

$\Longrightarrow \gamma(G_1) > g.$