12 Mar 2025

Random Graphs

Two (closely related) models.

Both are prob distribs over graphs with n vertices (for any pos. integer n).

Gilbert

$G(n,p)$: random graph where each pair $\{u,v\}$ is present in edge set independently with probability p.

$$\Pr\left(G=(V,E)\right) = p^{\#|E|} \cdot (1-p)^{\binom{n}{2}-\#|E|}$$

Erdos Renyi

$G(n,m)$: unif. random sample from $\{$graphs with m edges on vertex set $V\}$

$$\Pr\left(G=(V,E)\right) = \begin{cases} \dfrac{1}{\binom{\binom{n}{2}}{m}} & \text{if } \#|E|=m \\ 0 & \text{otherwise.} \end{cases}$$

The Erdos Renyi models

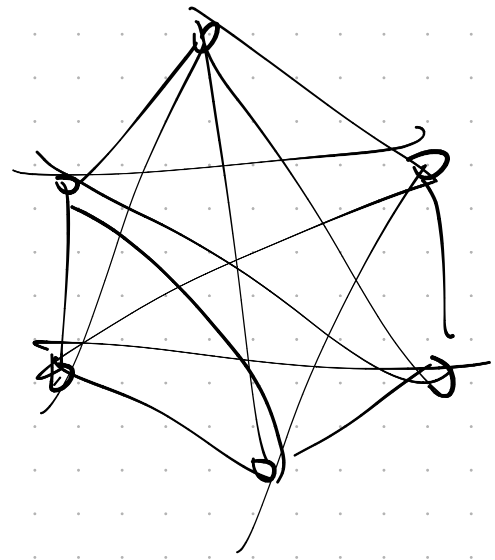
Evolving random network:

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_{\binom{n}{2}}$$

all have same n-element vertex set, V.

G_0 = empty graph.

$G_{i+1} = G_i$ + one additional random edge different from the edges of G_i.



Expected # of edges in $G(n,p)$?

$$p \cdot \binom{n}{2}$$

Chernoff $\Longrightarrow$ as soon as p large enough that $p \cdot \binom{n}{2} \gg 1$

$$\frac{(\text{actual \# of edges in } G(n,p))}{p \cdot \binom{n}{2}} \in [1-\varepsilon, 1+\varepsilon]$$

with probability $> 1 - 2e^{-\frac{1}{3}\varepsilon^2 p \binom{n}{2}}$

How large must p be to make $G(n,p)$ connected with high probability?

If $p \cdot \binom{n}{2} < (1-\varepsilon) \cdot (n-1)$ $\quad /\!\!/\!\!/ \quad p < \frac{1-\varepsilon}{2n}$

$G(n,p)$ probably disconnected.

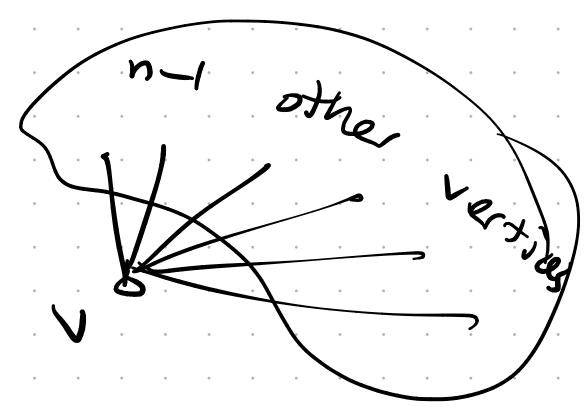
Obs. If $n > 1$ and G has an isolated vertex, then G must be disconnected.

Expected # isolated vertices in $G(n,p)$?

$$\boxed{n \cdot (1-p)^{n-1}}$$

vertices

$\Pr(v \text{ is isolated})$



when is this $\ll 1$?

$$1-p < e^{-p},$$

$$n(1-p)^{n-1} < n e^{-(n-1)p}$$

If $p \geq \frac{\ln(n) + c}{n-1}$ ... $\Pr(\exists \text{ isol vertex}) < \frac{1}{e^c}$

$$(n-1)p \geq \ln(n) + c$$

$$-(n-1)p \leq \ln(1/n) - c$$

$$e^{-(n-1)p} \leq \frac{1}{n} \cdot e^{-c}$$

$$n e^{-(n-1)p} \leq e^{-c},$$

When is $\mathbb{E}[\#\text{ isolated vertices}] >> 1$?

Ans. when $n(1-p)^{n-1} >> 1$.

Recall $1-p > e^{-p-p^2}$ for $0<p<\frac{1}{2}$.

So if $n \cdot e^{-(n-1)(p+p^2)} >> 1$

$\mathbb{E}[\#\text{ isolated vertices}] >> 1$.

when $p = \dfrac{\ln(n) - c}{n-1}$

$$p + p^2 = p(1+p)$$

$$(n-1)(p+p^2) = (n-1)p(1+p)$$
$$= (\ln(n) - c)\left(1 + \frac{\ln(n) - c}{n-1}\right)$$

...... (some algebra.....)

$\text{Var}(\#\text{ isol. vertices})$ ....

Let $X_u = \begin{cases} 1 & \text{if } u \text{ isolated} \\ \emptyset & \text{if } u \text{ not isolated.} \end{cases}$

$$\mathbb{E}[X_u] = (1-p)^{n-1}.$$

$\#$ isol vertices $= X = \sum_{u \in V} X_u$.

$$\mathbb{E}[X^2] - (\mathbb{E}X)^2$$

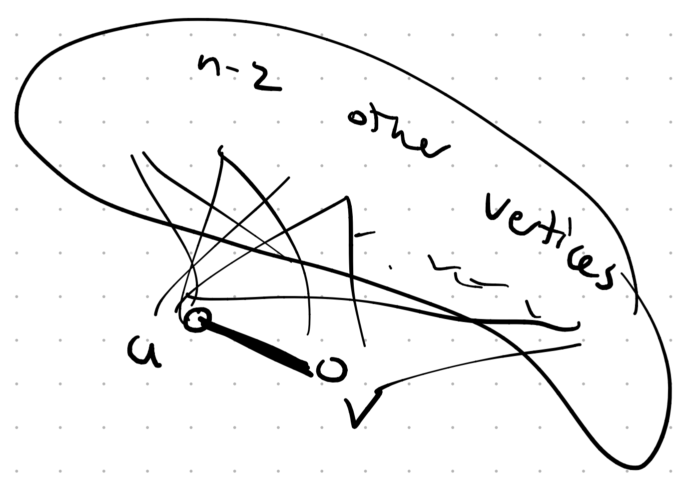
$$= \sum_u \sum_v \underbrace{\mathbb{E}[X_u X_v]}_{??} - \underbrace{\mathbb{E}[X_u]}_{(1-p)^{n-1}} \underbrace{\mathbb{E}[X_v]}_{(1-p)^{n-1}}$$

If $u = v$, $X_u X_v = X_u^2 = X_u$

$$\mathbb{E}[X_u^2] = (1-p)^{n-1}$$

If $u \neq v$

$$\mathbb{E}[X_u X_v] = (1-p)^{2n-3}.$$



$\mathrm{Var}(X)$

$$= \mathbb{E}[X^2] - \mathbb{E}(X)^2$$

$$= n \cdot (1-p)^{n-1} + n(n-1)(1-p)^{2n-3} - n^2 (1-p)^{2n-2}$$

$$= n(1-p)^{n-1} \left[1 + (n-1)(1-p)^{n-2} - n(1-p)^{n-1} \right]$$

$$= n(1-p)^{n-1} \left[1 + (n-1)(1-p)^{n-2} - n(1-p)(1-p)^{n-2} \right]$$

$$= n(1-p)^{n-1} \left[1 + (pn-1)(1-p)^{n-2} \right]$$

$$\mathrm{Var}(X) = E(X)\left[1 + (pn-1)(1-p)^{n-2}\right]$$

$$\Pr(\text{no isolated vertices})$$

$$\leq \Pr\left(|X - EX| \geq EX\right)$$

$$\leq \frac{\mathrm{Var}(X)}{(EX)^2} = \frac{1 + (pn-1)(1-p)^{n-2}}{n \cdot (1-p)^{n-1}}$$

Set $p \leq \frac{\ln n}{2n-2}$, do some algebra,

$\text{RHS} \leq n^{-0.3}$.

TL;DR. $p \leq \frac{1}{2} \cdot \frac{\ln n}{n}$

$\implies$ almost surely $\exists$ isol vtx and $G(n,p)$ disconnected

$p > \frac{\ln n + c}{n-1} \implies$ almost surely no isolated vertices

... but is it connected?