5 Feb 2025 Chernoff and Applications

Recap. (stated w/o proof last time)

$X_1, \ldots, X_n$ $[0,1]$-valued, indep't

$X = X_1 + \cdots + X_n$ $\qquad 0 \le \varepsilon \le 1$

$$\Pr(X \ge (1+\varepsilon)\mathbb{E}X) \le \exp\left(-\tfrac{1}{3}\varepsilon^2 \mathbb{E}[X]\right)$$

$$\Pr(X \le (1-\varepsilon)\mathbb{E}X) \le \exp\left(-\tfrac{1}{2}\varepsilon^2 \mathbb{E}[X]\right)$$

Moment gen fcn: $M_X(t) = \mathbb{E}[e^{tX}]$

Cumulant gen fcn: $K_X(t) = \ln M_X(t)$

$$M_X = \prod_{i=1}^{n} M_{X_i} \qquad K_X = \sum_{i=1}^{n} K_{X_i}.$$

$$M_X(t) = \sum_{n=0}^{\infty} \frac{\boxed{m_n(X)}}{n!} t^n$$

def'n of $m_n(X)$

" " $K_n(X)$

$$K_X(t) = \sum_{n=0}^{\infty} \frac{K_n(X)}{n!} t^n$$

$$e^{tX} = 1 + tX + \frac{t^2}{2!}X^2 + \cdots + \frac{t^n}{n!}X^n + \cdots$$

$$\mathbb{E}[e^{tX}] = 1 + \underbrace{\boxed{t\,\mathbb{E}X + \frac{t^2}{2!}\mathbb{E}[X^2] + \cdots + \frac{t^n}{n!}\mathbb{E}[X^n] + \cdots}}_{y=}$$

$\Longrightarrow$ $m_n(X) = \mathbb{E}[X^n]$. n^{th} moment of X.

$$\ln(1+y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \cdots$$

Expand, substituting $y = t\mathbb{E}X + \frac{1}{2}t^2\mathbb{E}X^2 + \frac{1}{6}t^3\mathbb{E}X^3 + \dots$

and ignoring t^4 and higher powers.

$$\begin{aligned}
\ln M_X(t) = \;& y && t\mathbb{E}X + \tfrac{1}{2}t^2\mathbb{E}X^2 + \tfrac{1}{6}t^3\mathbb{E}X^3 + \dots \\
& -\tfrac{1}{2}y^2 && -\tfrac{1}{2}\left[t^2(\mathbb{E}X)^2 + t^3(\mathbb{E}X)(\mathbb{E}X^2) + \dots\right] \\
& +\tfrac{1}{3}y^3 && +\tfrac{1}{3}t^3(\mathbb{E}X)^3 + \dots \\
& \dots &&
\end{aligned}$$

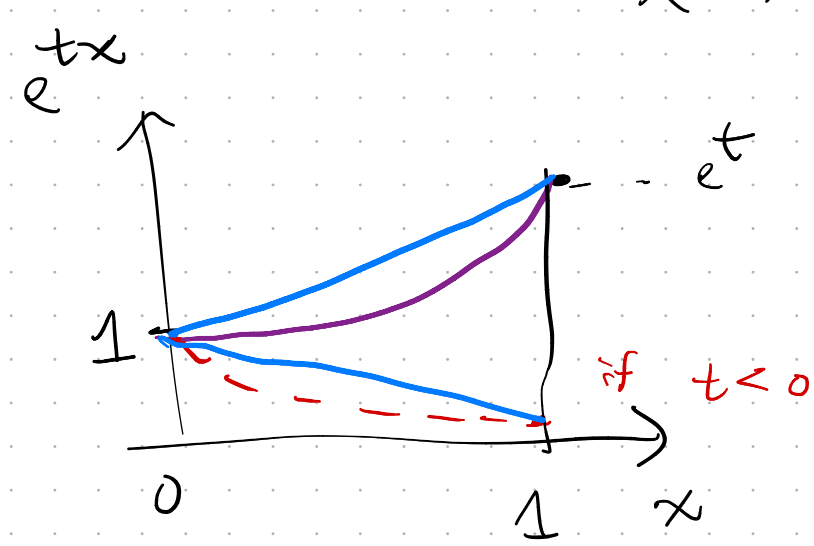
$$\underbrace{t\mathbb{E}X}_{K_1} + \frac{1}{2}t^2\Big[\underbrace{\mathbb{E}(X^2) - (\mathbb{E}X)^2}_{K_2}\Big] + \frac{1}{6}t^3\Big[\underbrace{\mathbb{E}X^3 - 3(\mathbb{E}X)(\mathbb{E}X^2) + 2(\mathbb{E}X)^3}_{K_3}\Big]$$

meaning $\Pr(X<0) = \Pr(X>1) = 0$.

Lem. If X is supported in $[0,1]$

then $\forall t$ $\quad M_X(t) \leq \exp\left((e^t - 1)\,\mathbb{E}[X]\right)$.

Proof.



$e^{tx} \leq (1-x) + e^t x$

for $0 \leq x \leq 1$.

$$M_X(t) = \mathbb{E}[e^{tX}] \leq 1 - \mathbb{E}X + e^t\,\mathbb{E}X$$

$$= 1 + (e^t - 1)\, \mathbb{E}X.$$

$$\leq \exp\left((e^t - 1)\, \mathbb{E}X \right). \qquad \blacksquare$$

<u>Chernoff Proof</u>. We have $X_1, \ldots, X_n$ taking values in $[0,1]$ and $X = X_1 + \ldots + X_n$.

So $\forall t$,

$$M_X(t) = \prod_{i=1}^{n} M_{X_i}(t) \leq \prod_{i=1}^{n} \exp\left((e^t - 1)\, \mathbb{E}X_i \right)$$

$$= \exp\left((e^t - 1) \sum_{i=1}^{n} \mathbb{E}X_i \right)$$

$$= \exp\left((e^t - 1)\, \mathbb{E}X \right)$$

Use Markov's Ineq.

if $t > 0$

$$\Pr\left(X \geq (1+\varepsilon)\mathbb{E}X \right) = \Pr\left(e^{tX} \geq e^{(1+\varepsilon)t\, \mathbb{E}X} \right)$$

$$\leq \underbrace{\mathbb{E}\left[e^{tX} \right]}_{M_X(t)} \cdot e^{-(1+\varepsilon)t\, \mathbb{E}X} \qquad \text{[Markov's]}$$

$$\leq e^{(e^t - 1)\mathbb{E}X} \, e^{-(1+\varepsilon)t\, \mathbb{E}X}$$

$$= \exp\left(\left(e^t - 1 - (1+\varepsilon)t \right) \mathbb{E}X \right)$$

Set $t = \ln(1+\varepsilon)$. $e^t - 1 - (1+\varepsilon)t$

$$= \varepsilon - (1+\varepsilon)\ln(1+\varepsilon)$$

$$\leq -\frac{1}{3}\varepsilon^2$$

for $0 < \varepsilon \leq 1$

using Taylor series.

$$\Pr(X \geq (1+\varepsilon)\mathbb{E}X) \leq \exp\left(-\frac{1}{3}\varepsilon^2 \, \mathbb{E}X\right).$$

Lower tail bound is done same way using

$t > 0$

$$\Pr(X \leq (1-\varepsilon)\mathbb{E}X) = \Pr\left(e^{-tX} \geq e^{-(1-\varepsilon)t\mathbb{E}X}\right)$$

... then steps look basically same as above. Set $t = -\ln(1-\varepsilon)$.

Back to balls in bins.

Focus on bin i with occupancy

$$X = L_i = X_1 + X_2 + \dots + X_m$$

$$X_j = \begin{cases} 1 & \text{if ball } j \text{ lands in bin } i \\ 0 & \text{if not.} \end{cases}$$

$\mathbb{E}X_j = \frac{1}{n}$, $\quad \mathbb{E}X = \frac{m}{n}$,

$$\Pr(X \geq (1+\delta)\mathbb{E}X)$$
$$\leq \exp\left(-\frac{1}{3}\delta^2 \frac{m}{n}\right)$$

$$\Pr(X \leq (1-\delta)\mathbb{E}X)$$
$$\leq \exp\left(-\frac{1}{2}\delta^2 \frac{m}{n}\right)$$

Recall: We wanted m to be large enough that

$$\Pr(|X - \mathbb{E}X| \geq \delta \cdot \mathbb{E}X) \leq \frac{1}{2n}$$

i.e., we're trying to solve

$$e^{-\frac{1}{3}\delta^2 (m/n)} + e^{-1/2\,\delta^2 (m/n)} \leq \frac{1}{2n}$$

$$\Big\Uparrow \text{ strengthen}$$

$$2e^{-1/3\,\delta^2 (m/n)} \leq \frac{1}{2n}$$

$$\frac{1}{3}\delta^2 (m/n) \geq \ln(4n)$$

$$m \geq 3n \ln(4n) / \delta^2$$

$$= 27n \ln(4n) / \varepsilon^2 .$$

Say we're analyzing an algorithm and there are N bad events we want to avoid.

We want: $\Pr(\text{any bad event happens}) \leq \frac{1}{2}$

$$\Uparrow \text{(union)}$$

each $\Pr(\text{bad event}) \leq \frac{1}{2N}$

Say each bad event is a sum of $[0,1]$ rand vars being ε-far from expectation.

$\underline{N=1}$: $\quad e^{-\frac{1}{3}\varepsilon^2 \mathbb{E}X} \leq \frac{1}{2} \Rightarrow \mathbb{E}X = \Omega(\varepsilon^{-2})$.

Same if you use Chebyshev!

"$\underline{N=n}$": Chebyshev would require

$$\mathbb{E}X = \Omega\left(\frac{n}{\varepsilon^2}\right)$$

Chernoff allows

$$\mathbb{E}X = \Omega\left(\frac{\ln n}{\varepsilon^2}\right).$$

"$N=2^n$": Chebyshev useless

needs $\mathbb{E}X = \Omega\left(\frac{2^n}{\varepsilon^2}\right)$

Chernoff allows $\mathbb{E}X = \Omega\left(\frac{n}{\varepsilon^2}\right)$.