CS 4850: Mathematical Foundations for the Information Age — March 13, 2009

Lecture Notes 24

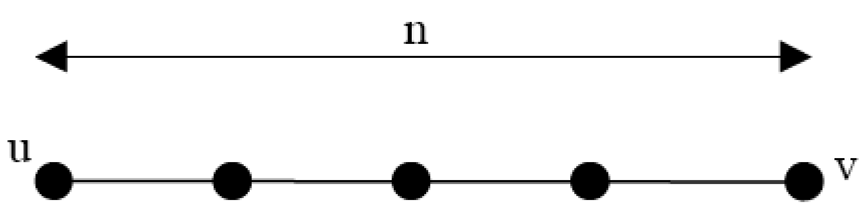
Professor: *John Hopcroft*

Scribe: *Di Wang (dw236)*
Hari Sankar (hss78)

Hitting Time

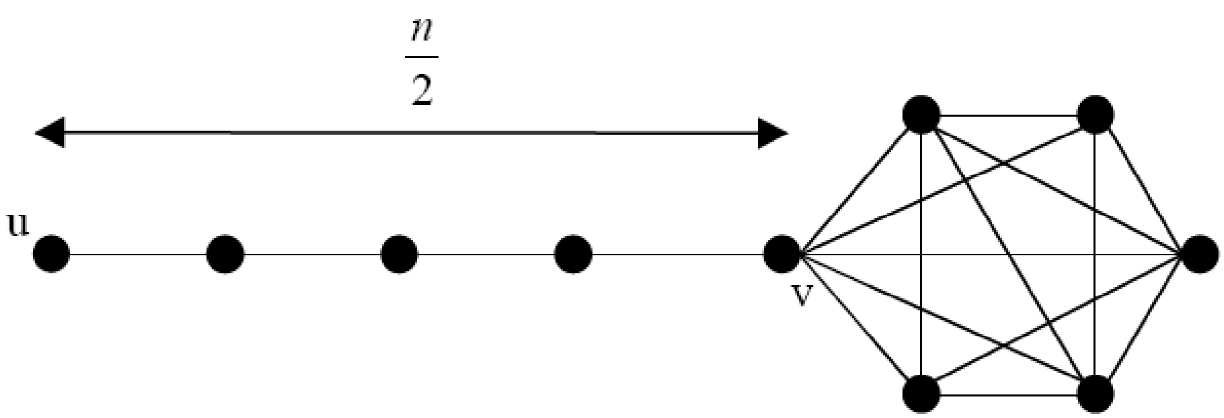
Definition: Hitting Time, h_{uv} is the expected time in a random walk to reach vertex v starting from vertex u.

Consider a graph without any cycles between u and v and n edges between them as shown in the figure below.



Hitting time, h_{uv} for the above situation $=O(n^2)$

Now we add some edges from vertex v in order to insert a clique of size n/2.This is shown in the figure below.



The vertex v has degree n/2-1(v itself is an edge of the clique).Suppose we start a random walk from v to u, there is a high probability that the clique is entered .Once this happens, it is going to take at an average O(n) steps to get out of the clique.

Hence, the hitting time, h_{vu} for the above situation = $O(n^3)$

However, in above figure, the hitting time is not symmetric.

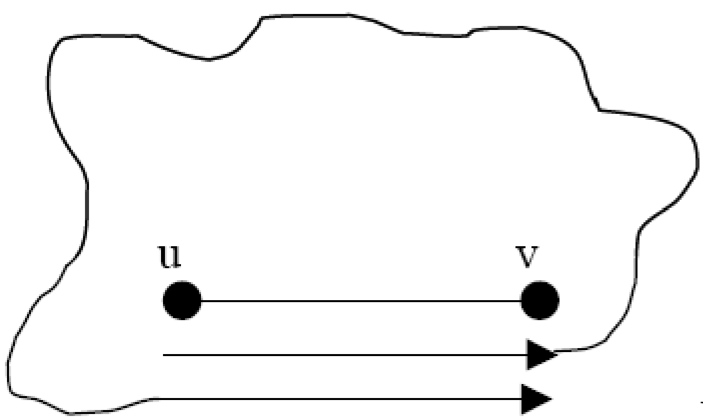
Hitting time, h_{uv} for the above situation = $O(n^2)$

What is the maximum possible hitting time in a graph of n vertices?

Lemma: If u and v are connected by an edge then hitting time, $h_{uv}+ h_{vu} \leq 2m$ where m is the number of edges in the graph.

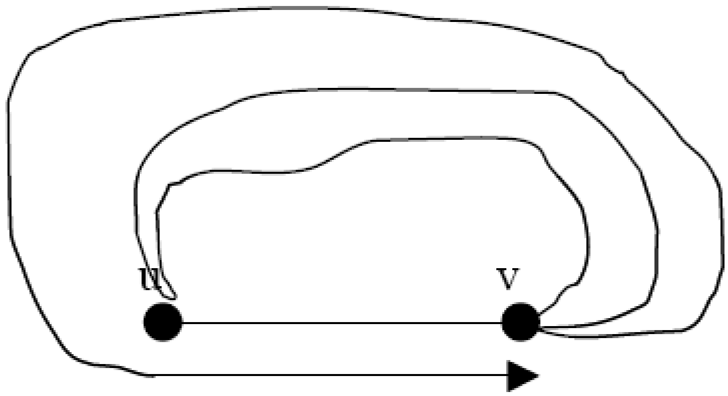
Proof: Since the probability of traversing any edge is equally likely,the probability of traversing an edge in any direction is 1/2m.

Therefore, it follows that the time between traversals of an edge is 2m.This is illustrated in the diagram below.



Expected Path Length = 2m

Given that the random walk in the above diagram traversed the edge uv, the expected time until next traversal of edge uv is 2m.Since random walks are memoryless, we can drop the condition. I.e. the path taken by the random walk could have entered u and gone to v multiple times (via different edges) before actually taking the edge uv. This situation is shown below.



Expected Path Length = 2m

Since the total path length is expected to be 2m, the path may go to u and come back to v in less than 2m steps. Hence it is proved that $h_{uv} + h_{vu} \le 2m$

What happens if there was no edge between u and v?

There is a path from u to v such that path length ≤n.From the previous lemma, it takes 2m steps for the traversal of the same edge. Hence hitting time, $h_{uv} \le 2mn \le n^3$

Commute Time

Definition: Commute(u, v) = expected time starting at u to reach v and then return to u .

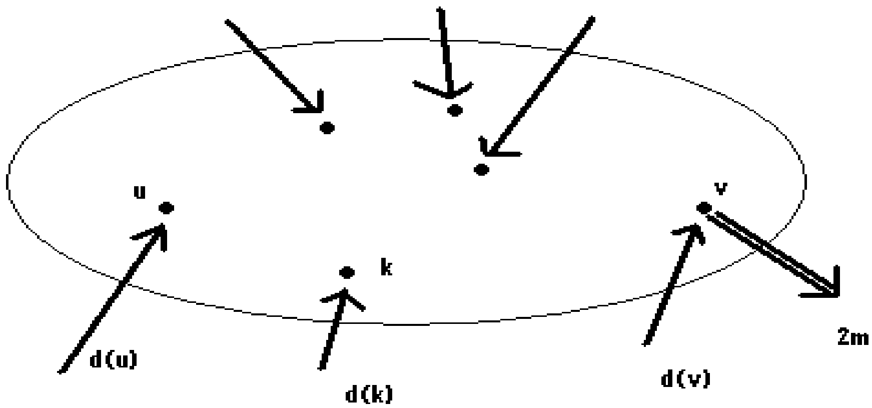
Theorem: Consider undirected graph with each edge replaced by one ohm resistor,

$$\text{Commute}(u, v) = 2mr_{u,v}$$

where $m = \#$ edges , $r_{u,v} =$ effective resistance between u and v .

Proof

Insert current at every node of value equal to the degree of the node, extract current at v.



$$\text{current extracted at } v = \sum_i d(i) = 2m$$

Look at voltages relative to node v, V_{uv}.

Because of current conservation,

$$d(u) = \sum_{k \text{ adj. to } u} (V_{uk} - V_{kv}) = \sum_k V_{uk} - \sum_k V_{kv}$$
$$= V_{uk}\, d(u) - \sum_k V_{kv}$$

Solve the above equation for V_{uv}

$$V_{uv} = \frac{1}{d(u)}\left(d(u) + \sum_k V_{kv}\right)$$
$$= \frac{1}{d(u)} \sum_k (1 + V_{kv})$$

Recall the hitting time of u and v

$$h_{uv} = \frac{1}{d(u)} \sum_k (1 + h_{kv})$$

We get $V_{uv} = h_{uv}$ (1) when current extracted at v.

By symmetry, when we extract current at u, we will get

$$\overline{V_{vu}} = h_{vu}$$

Then in the above case, when we reverse all current such that we insert current at u, and extract current at every node, we will get

$$\overline{\overline{V_{vu}}} = -\overline{V_{vu}} = -h_{vu}$$
$$\overline{\overline{V_{uv}}} = -\overline{\overline{V_{vu}}} = h_{vu} \qquad (2)$$

By superposition of current in (1) and (2), we get

$$V_{uv} + \overline{\overline{V_{uv}}} = h_{uv} + h_{vu}$$

The LHS of the above equation is the resulted voltage of u relative to v in the superposed network, where current of value 2m enter the network at node u, and flows out at node v, so

$$Commute(u,v) = h_{uv} + h_{vu} = voltage = current*resistance = 2mr_{uv}$$

Corollary: For any vertices u and v in an connected n-vertex graph,

$$Commute(u,v) < n^3$$

Proof

Since graph connected, there exists a path from u to v, and the path is of length at most n.

Then the resistance of this path is less than n, which means the effective resistance between u and v is less than n. By our theorem, $Commute(u,v) = 2mr_{u,v} < 2mn \le 2C(n,2)n < n^3$.