14 March 2025

Boolean Satisfiability, P, and NP

Plan

* Boolean Satisfiability Problem
* Announcements
* Defining P vs. NP.

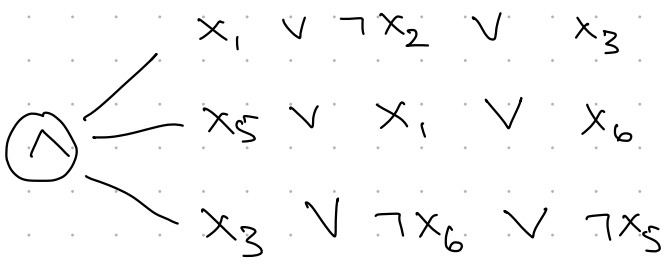
Systems of Boolean Equations

XOR-SAT

$$X_1 \oplus \neg X_2 \oplus X_3$$
$$X_5 \oplus X_1 \oplus X_6$$
$$X_3 \oplus \neg X_6 \oplus \neg X_5$$

} Does there exist an assignment $\vec{x} = \vec{a} \in \{0,1\}^n$ s.t. every expression is 1?

CNF-SAT (i.e. AND-of-OR SAT)



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XOR-SAT ← Polynomial-time solver! (via Gaussian Elimination)

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$\wedge$
- $x_1 \vee \neg x_2 \vee x_3$
- $x_5 \vee x_1 \vee x_6$
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CNF-SAT (i.e. AND-of-OR SAT)

$$\bigwedge \begin{cases} x_1 \vee \neg x_2 \vee x_3 \\ x_5 \vee x_1 \vee x_6 \\ x_3 \vee \neg x_6 \vee \neg x_5 \end{cases}$$

} Does there exist an assignment $\vec{x} = \vec{a} \in \{0,1\}^n$ s.t. every clause is satisfied?

No known efficient algorithm!

Boolean (CNF) Formula

$$\phi = (x_1 \vee \neg x_2 \vee x_3) \wedge (x_5 \vee x_1 \vee x_6) \wedge (x_3 \vee \neg x_6 \vee \neg x_5)$$

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n variables

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SAT. Given a boolean formula ϕ, does there exist a satisfying assignment?

That is, $\vec{x} = \vec{a} \in \{0,1\}^n$ s.t. $\phi(\vec{a}) = 1$

$$\phi = (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_1) \wedge (x_3 \vee x_1)$$

Is ϕ satisfiable?

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Is ϕ satisfiable?

x_1	x_2	x_3	C_1	C_2	C_3	ϕ
0	0	0				
1	0	0				
0	1	0				
0	0	1				
1	1	0				
1	0	1				
0	1	1				
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Announcements

* HW5 ongoing

* Prelim 2 Conflict Survey on Ed.

* Recitation on Saturday
  ↳ Mathematical Algorithms

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Observation. Given an assignment $\vec{x} = \vec{a}$ there is an efficient algorithm to verify

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For each clause $C = (\ell_1 \vee \ell_2 \vee \cdots \ell_k) \in \phi$.

| Assign literals $\ell_i \in \{x_i, \neg x_i\}$ according to a_i
| if $C(a) = 0$, Return False

Return True

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Observation. Given an assignment $\vec{x} = \vec{a}$ there is an efficient algorithm to verify $\phi(\vec{a}) = 0$ or $\phi(\vec{a}) = 1$

SOLVING SAT

⇓

Brute Force SAT Algorithm

For all assignments $\vec{a} \in \{000\cdots0, 000\cdots1, \cdots, 111\cdots1\}$

  Verify $(\phi, \vec{a})$

  If $\phi(\vec{a}) = 1$: Return True

Return False

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} $O(n\,m)$

- Set each literal in each clause
- Determine if there is a 1 in each clause.

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Brute Force SAT Algorithm

For all assignments $\vec{a} \in \{000\cdots0, 000\cdots1, \ldots, 111\cdots1\}$ ($= 2^n$)

  Verify $(\phi, \vec{a})$ } $O(nm)$

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- Set each literal in each clause
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$\Rightarrow \Theta(2^n \cdot \text{poly}(n, m))$

Theorem. There exists a $O(2^n \cdot \text{poly}(n,m))$ - time algorithm for CNF-SAT.

Better Algorithms?

Not yet ---

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No known $O(1.999^n)$ time algorithm for SAT

But why?

Complexity Theory

Categorize problems based on how EASY / HARD they are to solve.

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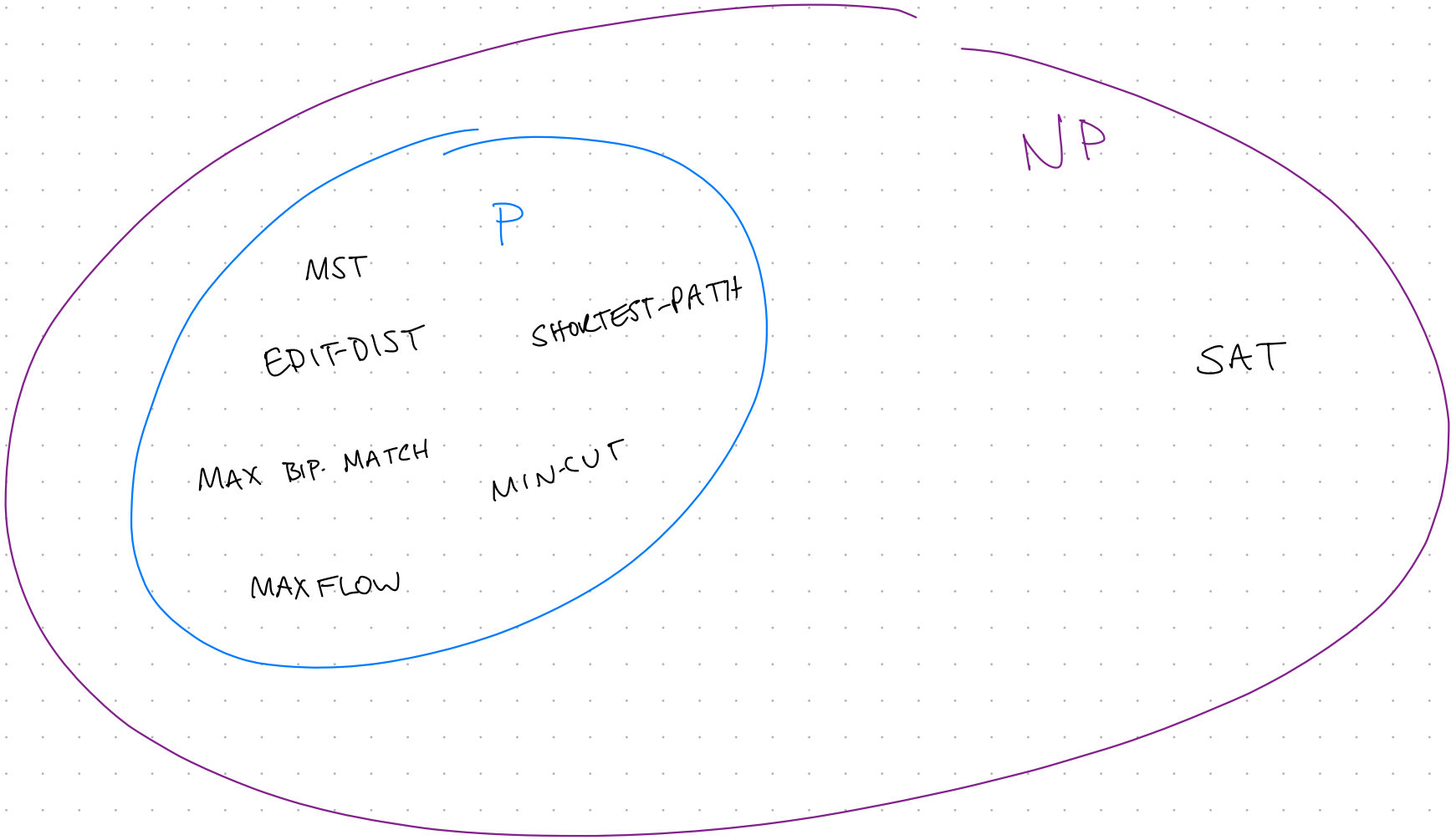
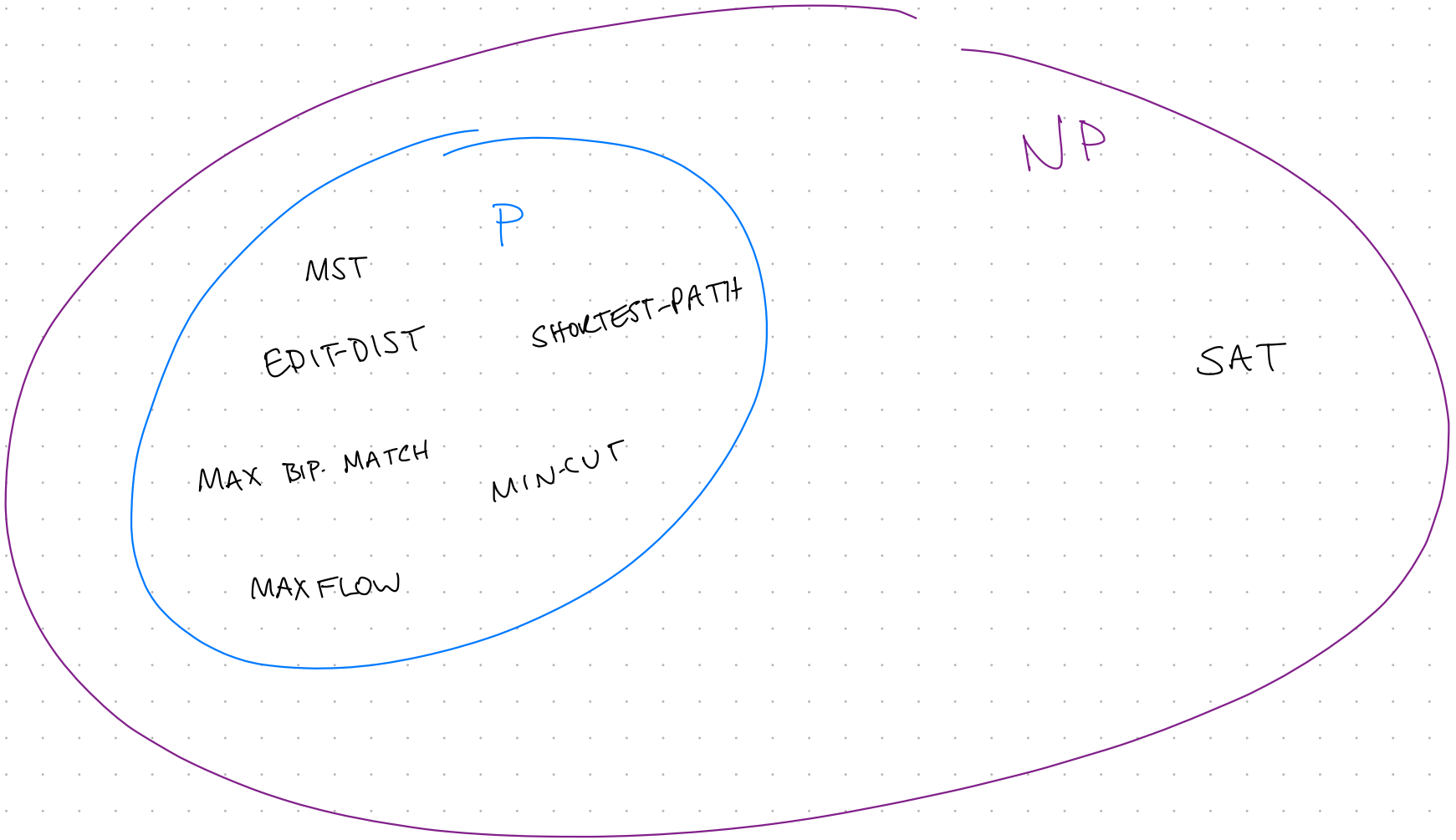
P vs. NP. Does efficient verification imply efficient solving?

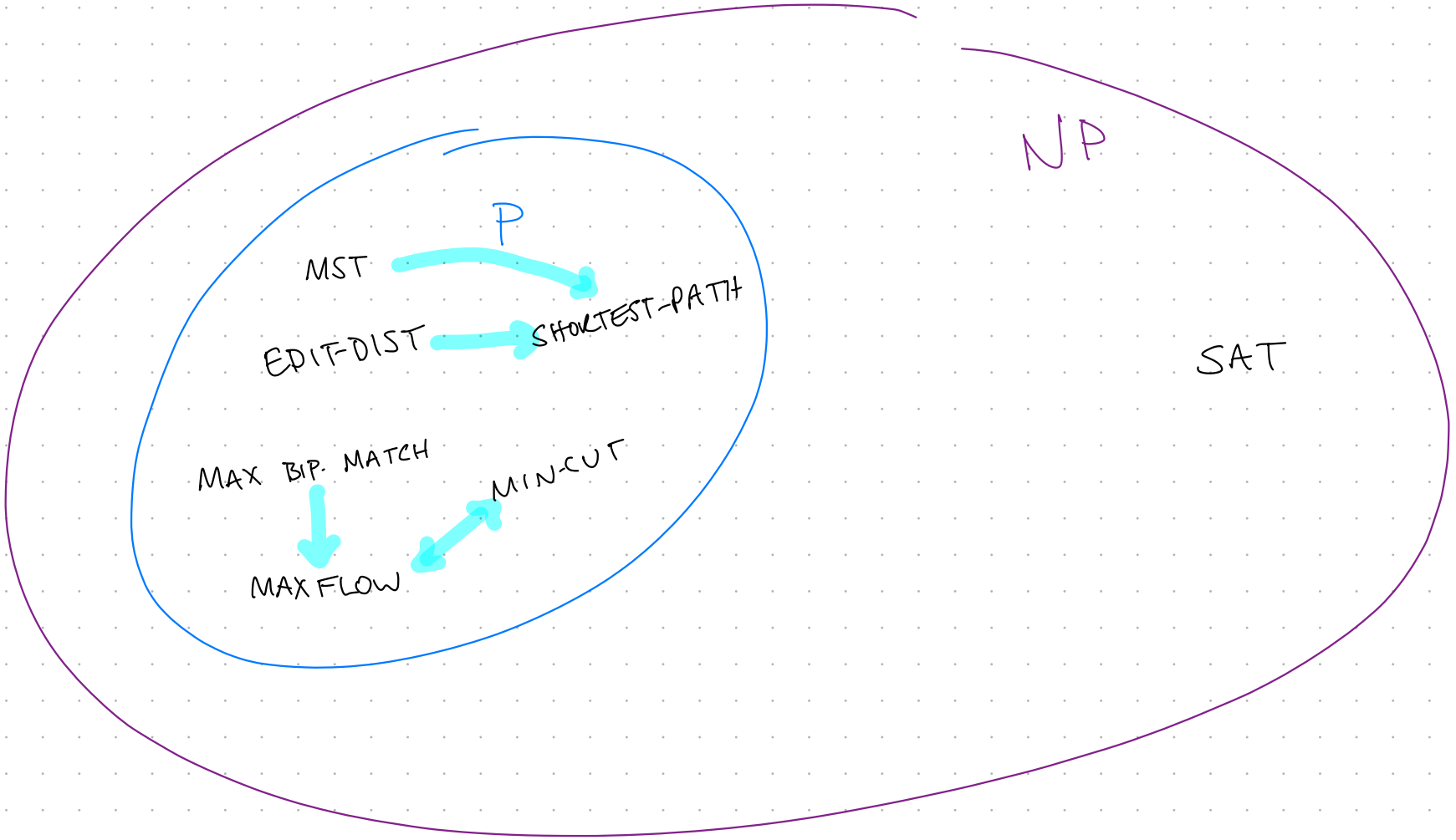

NP
P
MST
EDIT-DIST
SHORTEST-PATH
SAT
MAX BIP. MATCH
MIN-CUT
MAX FLOW



Reductions as a measure of hardness

Suppose Q reduces to R via poly-time reduction.

$$Q \leq_p R$$



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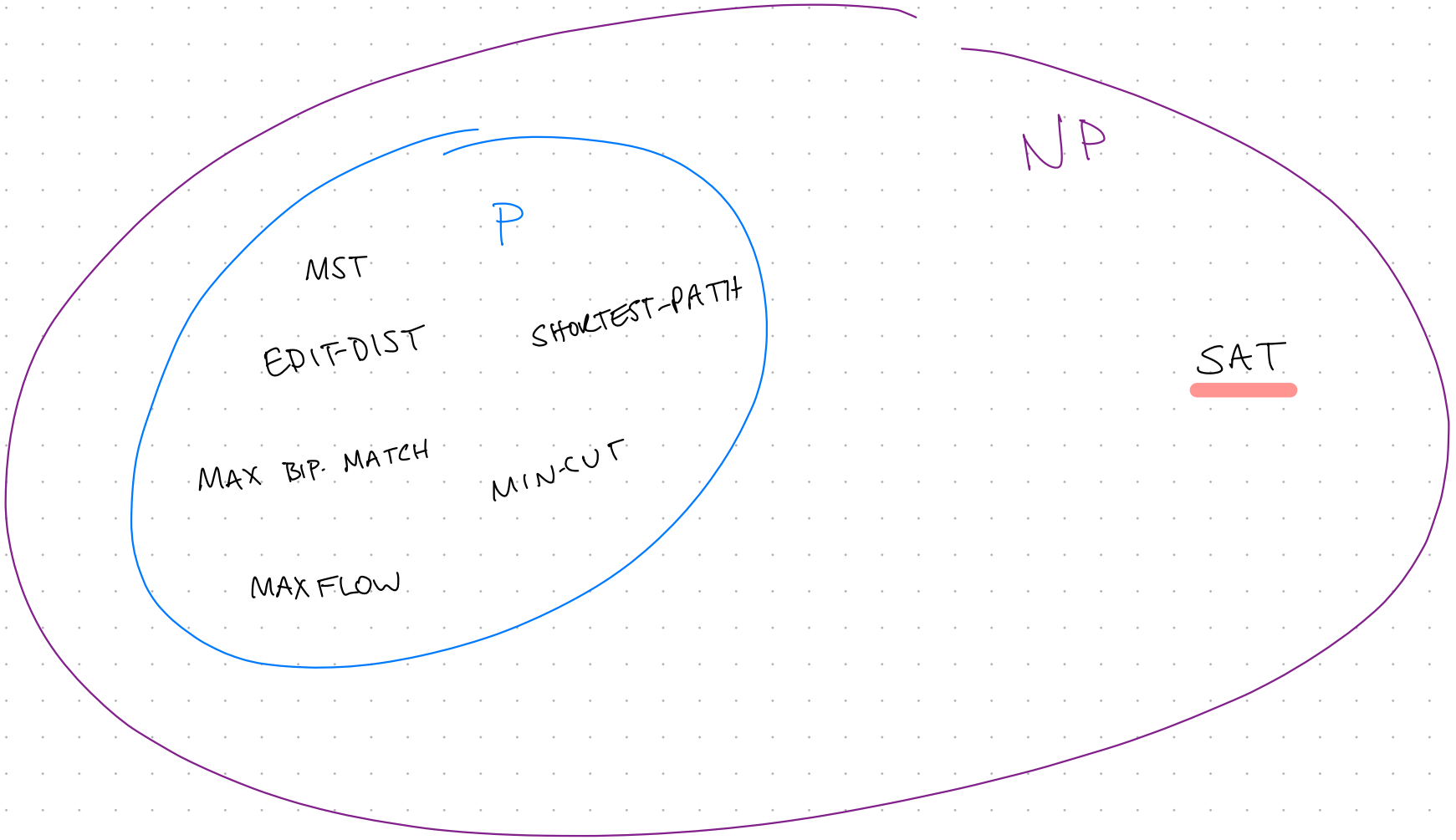

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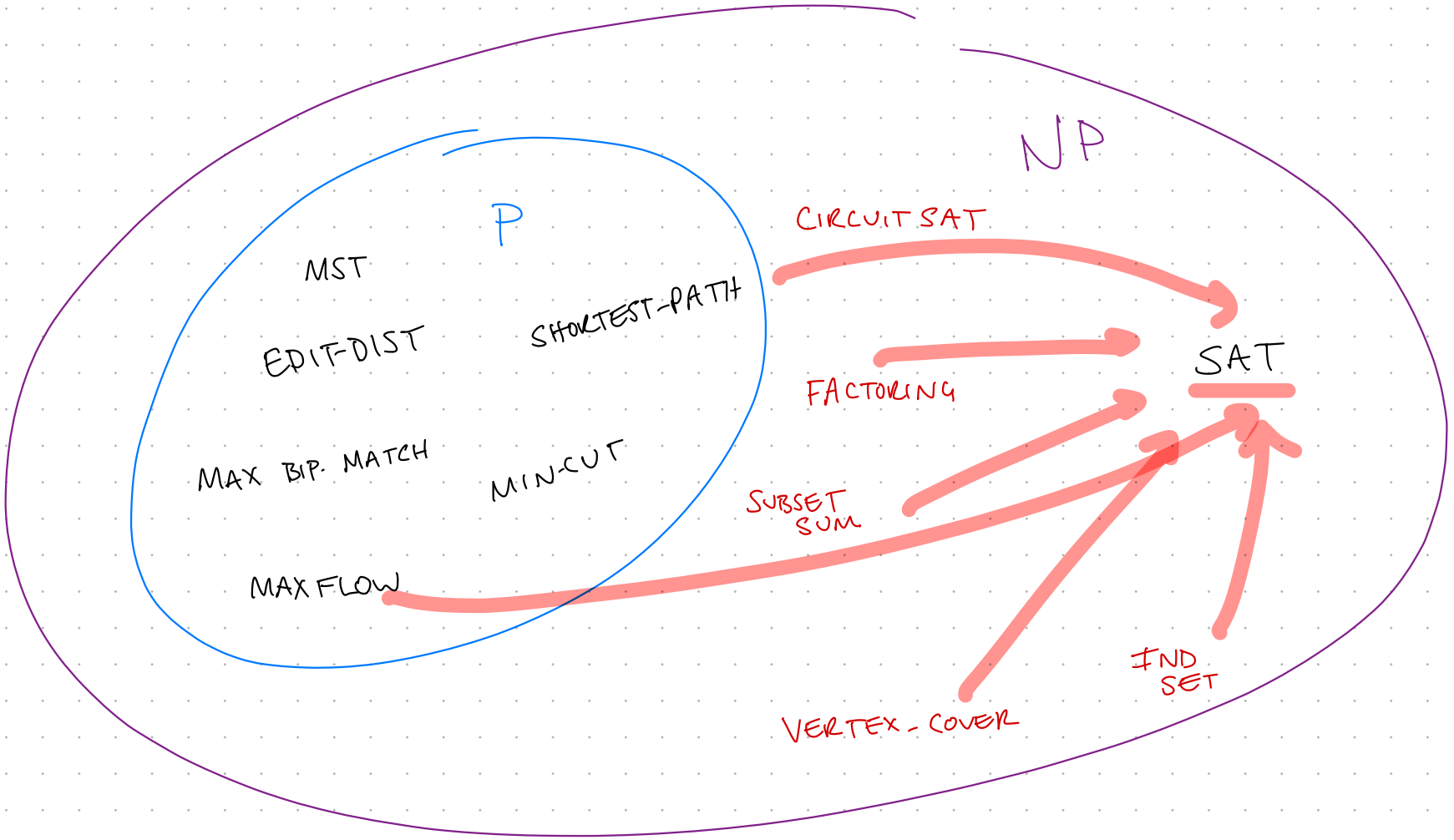

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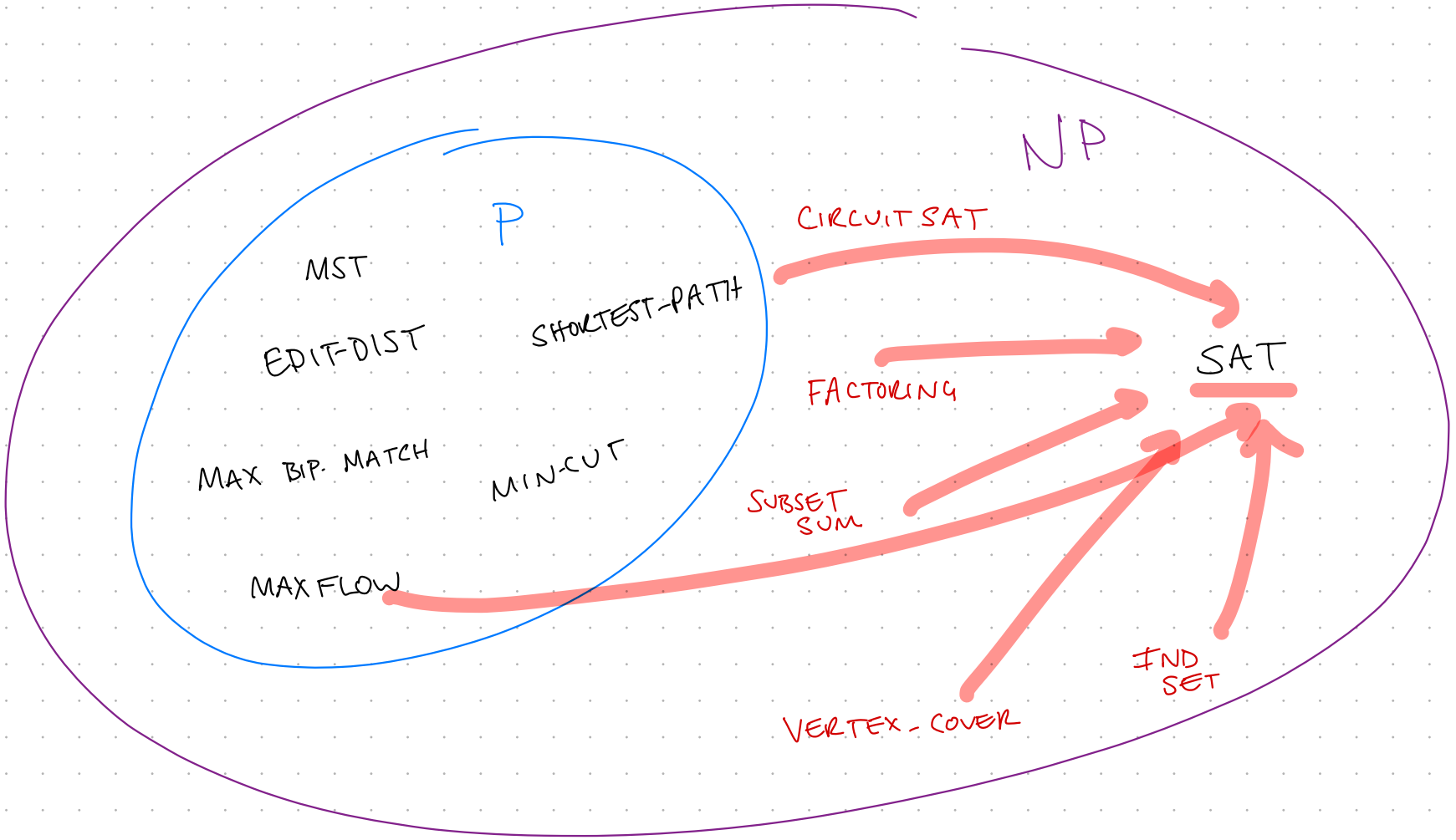
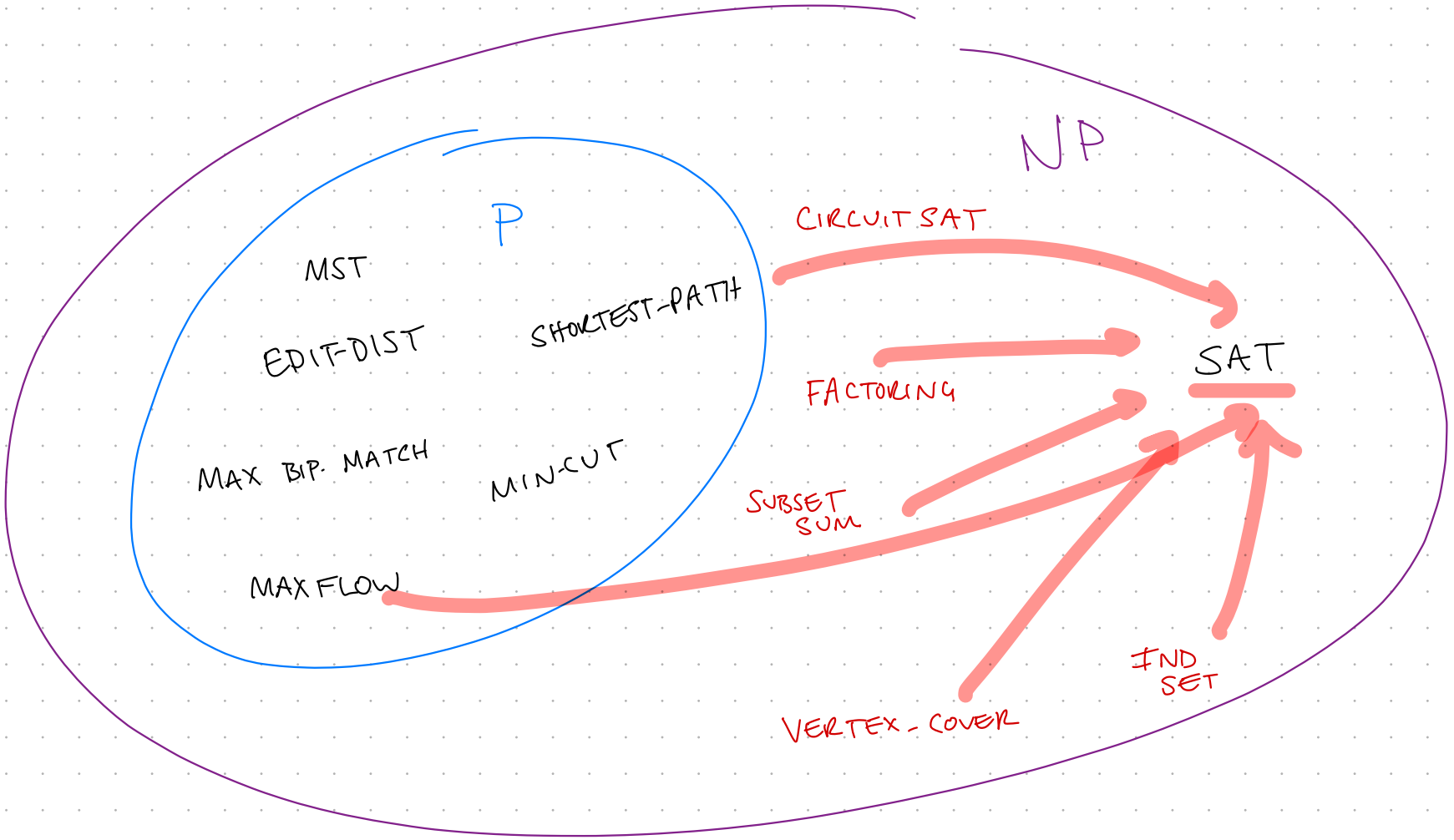


Theorem. Every problem $Q \in NP$ reduces to SAT!



Theorem. Every problem $Q \in NP$ reduces to SAT!

If any efficiently-verifiable problem is hard to solve, then SAT is hard to solve.

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NP-Hard ≡ Q is NP-Hard if every problem in NP reduces to Q.

NP-Complete

Q is NP-Complete if.

* $Q \in NP$
* Q is NP-Hard

The Cook-Levin Theorem SAT is NP-Complete.