12 February 2025

Stable Matching

Plan

* Stable Matching Problem
* Announcements
* Gale-Shapley Algorithm

Problem. Medical Residency Job Market.

* Doctors apply to Residency Programs.

* Doctors have preferences over Residencies
  ↳ prestige, rigor, location, etc...

* Residencies have preferences over Doctors
  ↳ grades, letters of rec, contributions to student body

Classic Employment Market

* Residencies "bid" on Doctors via Salaries
  ↳ Decentralized / Asynchronous

If Doctor d receives offer from Residency r
should she accept offer?

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Leads to Negotiations / Strategizing

Alternative: Matching Market

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Today's Simplifying Assumption

* Equal # of Doctors & Residencies
* Every Residency hiring exactly one Doctor

Outputs a "Matching" of Doctors $\longleftrightarrow$ Residencies

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<u>Defn</u>. Given a collection of Doctors D, Residencies R

a <u>matching</u> is a set of pairs M s.t.

↳ Every pair $(d, r) \in M$ consists of <u>exactly</u> one $d \in D$ and one $r \in R$.

↳ Every Doctor / Residency involved in <u>at most</u> one pair in M.

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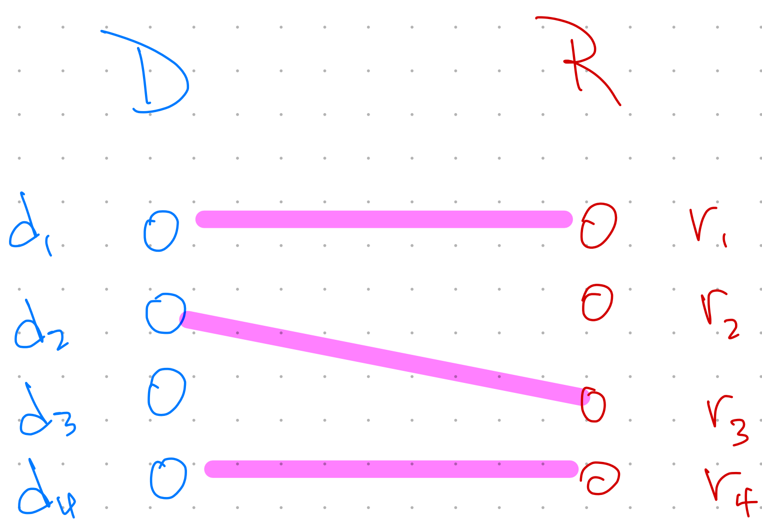
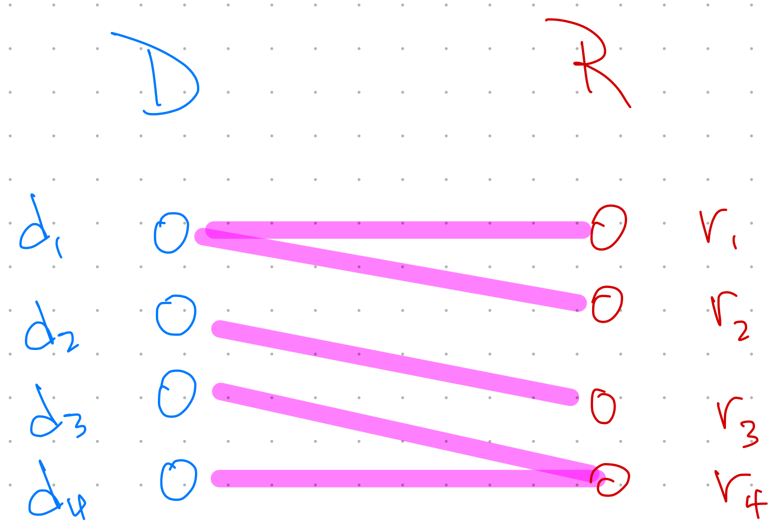


Matching



Not a matching

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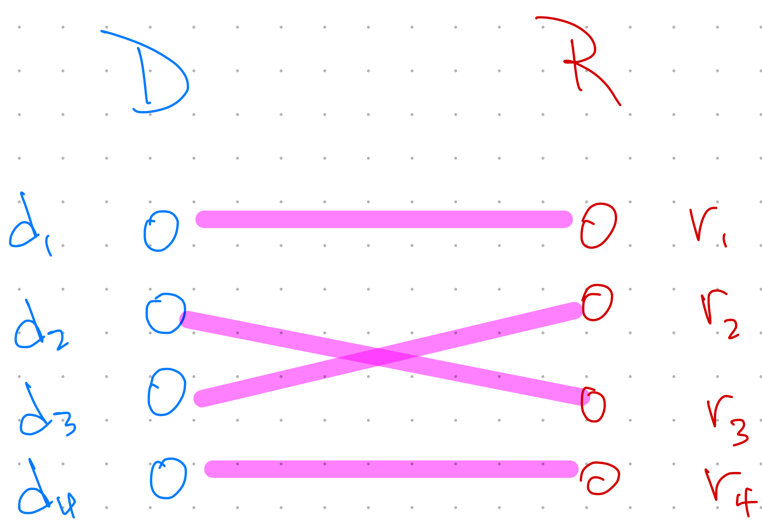

Matching

A <u>perfect matching</u> is a matching where every $d \in D$ and $r \in R$ is involved in <u>exactly</u> one pair in M.

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But why would Doctors / Residencies agree to enter the market?

Announcements.

* HW 2 due
  ↳ Solutions Released ASAP ~12 noon.

* Prelim 1. Thurs 7:30 - 9p.
  ↳ A-D Statler 196
  ↳ E-Z Statler 185

* Ed Shutdown
  ↳ Closing Discussions ~24hr prior to exam
  ↳ Reopens After Feb Break.

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But why would Doctors / Residencies agree to enter the market?

Why would a Doctor be upset with matching?

Why would a Residency be upset w/ matching?

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↳ Doctor d matched to Residency r,
but d prefers r' to r.

Why would a Residency be upset w/ matching?

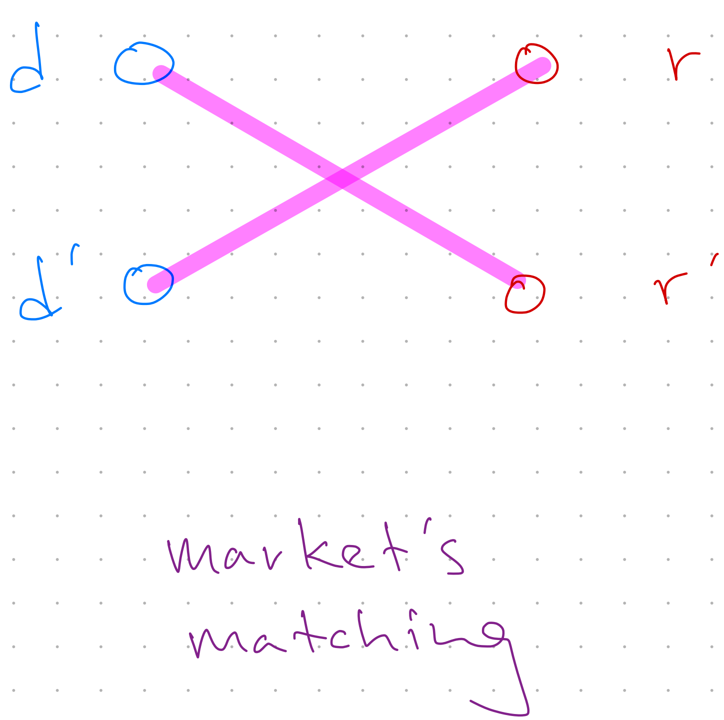
↳ Doctor d matched to Residency r,
but r prefers d' to d.

* Suppose Market outputs (d, r') and (d', r).

Defn. Instability

d prefers r to matched residency r', AND

r prefers d to matched doctor d'.



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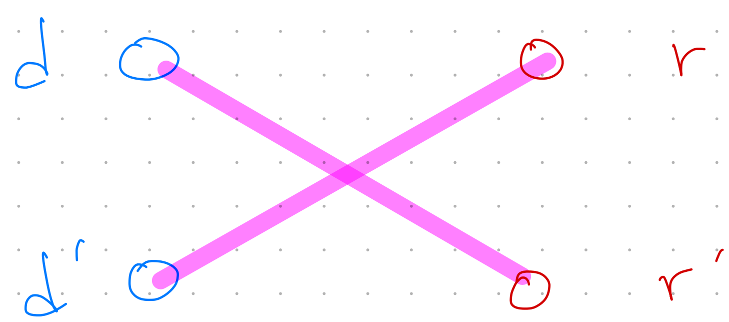

market's matching

d and r have incentive to reject matching

Defn. A perfect matching M is stable if it contains no instabilities:

$\forall d \in D \quad \forall r \in R$

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$d_1: r_1 > r_2 > r_3$ $\qquad$ $r_1: d_2 > d_1 > d_3$

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* When does a stable matching exist?

* If such a matching exists can we find it efficiently?

Answer. A stable matching always exists!

Theorem (Gale-Shapley).

Fix a collection of Doctors D & Residencies R with any set of preferences.

There is an efficient algorithm that returns a stable matching M between D & R.

Idea:

* Each residency maintains list of Doctors according to their prefences.

$$r: d_5 \succ d_3 \succ d_7 \succ d_1 \succ d_6 \succ d_2 \succ d_4$$

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* Each residency maintains list of Doctors according to their prefences.

* Iteratively make offers to next most-preferred doctor

$$r: \cancel{d_5} \succ \cancel{d_3} \succ \cancel{d_7} \succ d_1 \succ d_6 \succ d_2 \succ d_4$$

already made offers

next most-preferred

Gale-Shapley (D, R)

Initialize $M \leftarrow \emptyset$.

While $\exists$ $r \in R$ that hasn't made offers to every $d \in D$

* r makes offer to next most-preferred d

Output M

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* r makes offer to next most-preferred d

  if d is unmatched,

  $\hookrightarrow$ $M \leftarrow M \cup \{(d, r)\}$

  else d is matched to some $r' \in R$.

  if d prefers r to r'

  $\hookrightarrow$ $M \leftarrow M \setminus \{(d, r')\} \cup \{(d, r)\}$

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Some Questions

* While loop termination: Does GS always return?

* Matching: is M a perfect matching?

* Stability: is M stable?

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Every iteration involves offer from $r \in R$ to NEW $d \in D$.

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Iterations "indexed" by (r, d) pairs

$\leq |R| \cdot |D| = n^2$ iterations

Some Questions

* While loop termination: Does GS always return?

* Matching: is M a perfect matching?

* Stability: is M stable?

Key Observation (*)

Once $d \in D$ receives offer, d remains matched.

If d is re-matched from (d, r') to (d, r)

then $r \succ_d r'$.

if d is matched to some $r' \in R$.

if d prefers r to r'

$\hookrightarrow M \leftarrow M \setminus \{(d, r')\} \cup \{(d, r)\}$

re-matched

* Matching : is M a perfect matching?

YES

Proof.

GS Terminated

$\Longrightarrow \exists r \in R$ made offer to every $d \in D$.

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$\Longrightarrow$ $\exists r \in R$ made offer to every $d \in D$.

By Observation (*)

Every $d \in D$ received offer

$\Longrightarrow$ Every $d \in D$ must be matched.

$\Longrightarrow$ M is a perfect matching. $\blacksquare$

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(Residencies make offers in order of preference, made offer to d', but not d.)

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So $r' \succ_d r \Longrightarrow d' \succ_{r'} d$ No INSTABILITY. ∎