27 March 2024 — Boolean Satisfiability (SAT)

Plan

* Recall CNF-SAT
* Announcements
* Reductions To SAT
  - ↳ IND SET $\leq_p$ CNF-SAT
  - ↳ CIRCUIT-SAT $\leq_p$ 3SAT

Boolean Satisfiability

Given: boolean formula φ in Conjunctive Normal Form

CNF $\varphi = (x_1 \vee \neg x_2 \vee x_5) \wedge (x_2 \vee x_3 \vee x_7 \vee \neg x_6)$
$\wedge \ldots \wedge (\neg x_1 \vee \neg x_{100})$

Question:

Does there exist an truth assignment to $(x_1 \ldots x_n)$ that satisfies φ ?

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Given: boolean formula φ in Conjunctive Normal Form

CNF $\varphi = (x_1 \vee \neg x_2 \vee x_5) \wedge (x_2 \vee x_3 \vee x_7 \vee \neg x_6) \wedge \dots \wedge \underbrace{(\neg x_1 \vee \neg x_{100})}_{\text{clauses}}$

literals (variable or negation)

Question:

Does there exist an <u>truth assignment</u> to $(x_1, \dots, x_n)$ that <u>satisfies</u> φ ?

$\vec{a} = (1, 0, 0, \dots, 1)$

$\varphi(\vec{a}) = T$

Why study SAT?

Cook-Levin Theorem. SAT is NP-Complete.

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↳ SAT is in NP.

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Cook-Levin Theorem. SAT is NP-Complete.

↳ SAT is in NP.

Poly-time verifier for SAT.

$V(\varphi, \vec{a})$.

For each clause in φ

if $\vec{a}$ does not satisfy clause

Return $\perp$

Return ✓

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- ↳ SAT is in NP.
- ↳ SAT is NP-Hard.

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Every efficiently verifiable problem reduces to SAT!

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Every efficiently verifiable problem reduces to SAT!

Solving SAT is hard!

Solving SAT is powerful!

Announcements

* HW 7 due Thurs, 11:59 pm
  ↳ See Ed Post about update to Q3 (makes problem easier)

* Prelim Conflict Survey
  ↳ Link on Ed

* Prelim #2 Review Session
  ↳ April 9, 7-9 pm, Gates G01.

Every efficiently verifiable problem reduces to SAT!

Solving SAT is hard!

Solving SAT is powerful!

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Solving SAT is powerful!

Practical Algorithm design paradigm:

- Reduce problem to SAT
- Use optimized SAT SOLVER to solve the problem

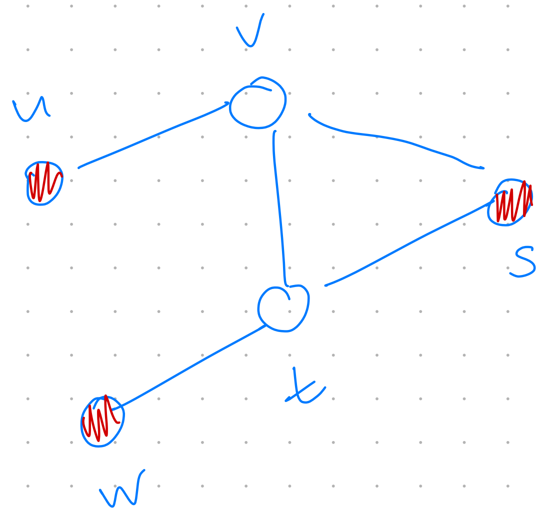
INDEPENDENT SET REDUCES TO CNF-SAT

↳ Given: Graph G, parameter k

$|S| \geq k$

Find: A subset $S \subseteq V$ of vertices such that no two vertices $u, v \in S$ share an edge $(u,v) \in E$

$G =$



$\xrightarrow{\text{reduction}}$ φ_G

φ_G satisfiable

$\Longleftrightarrow$

G has INDSET $\geq k$

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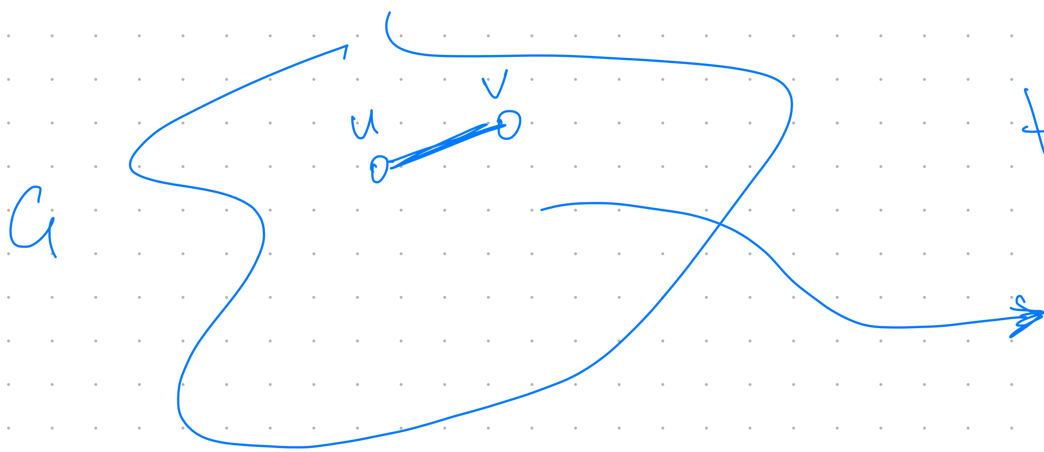

$\forall (u,v) \in E$

$\neg(u \in S) \vee \neg(v \in S)$

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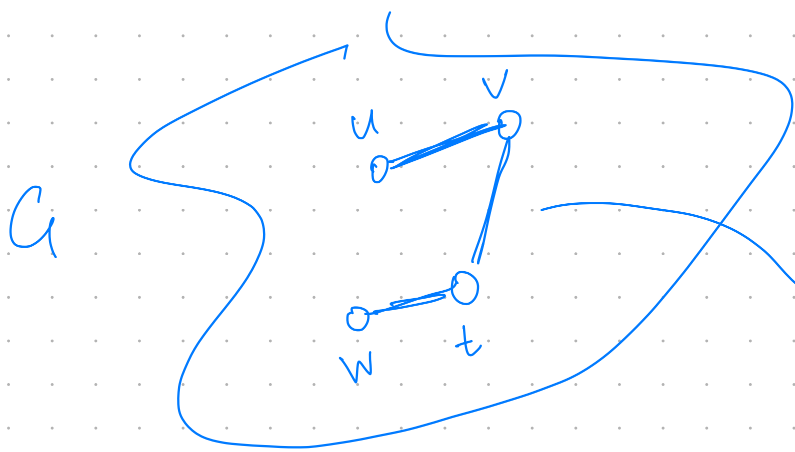


$\forall (u,v) \in E$

$$\neg\underbrace{(u \in S)}_{x_u} \vee \neg\underbrace{(v \in S)}_{x_v}$$

$$(\neg x_u \vee \neg x_v) \wedge (\neg x_v \vee \neg x_t) \wedge (\neg x_w \vee \neg x_t) \wedge \ldots \quad \forall e \in E$$

INDEPENDENT SET REDUCES TO CNF-SAT

↳ Given: Graph G, parameter k

Find: A subset $S \subseteq V$ of vertices $|S| \geq k$

such that no two vertices $u, v \in S$

share an edge $(u,v) \in E$

Need some mechanism to count to k!

Otherwise $S = \emptyset$, i.e. $\vec{a} = \vec{0}$ satisfies φ_G

$$(\neg x_u \vee \neg x_v) \wedge (\neg x_v \vee \neg x_t) \wedge (\neg x_w \vee \neg x_t) \wedge \ldots \quad \forall e \in E$$

$x_u^i = T \iff u$ is the i^{th} vertex in S

$$\overbrace{\boxed{x_u^1} \, \boxed{\dots} \, \boxed{x_u^i} \, \boxed{\dots} \, \boxed{x_u^k}}^{x_u}$$

Ind Set instance G, k

k boolean variable per vertex

$x_u^i = T \iff u$ is the i^{th} vertex in S

$$\overbrace{\boxed{x_u^1} \;\boxed{\cdots}\; \boxed{x_u^i} \;\boxed{\cdots}\; \boxed{x_u^k}}^{x_u}$$

k boolean variable per vertex

1

u is "in" S at most once

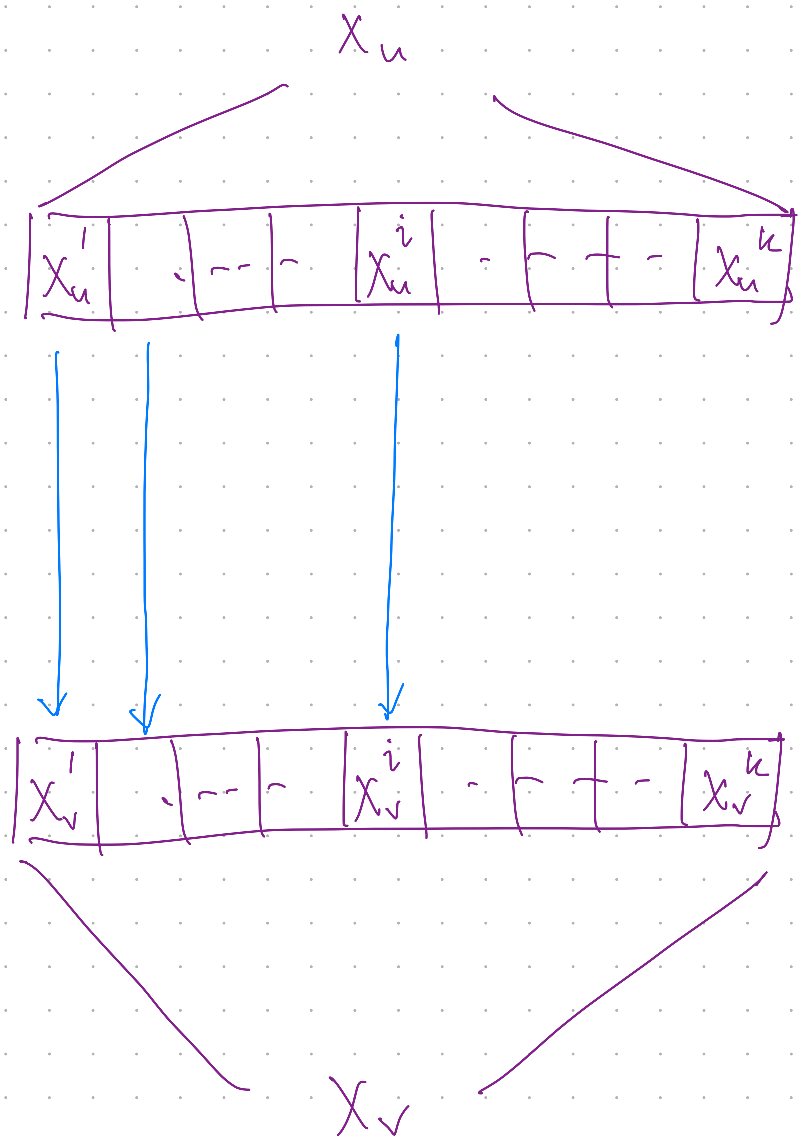
For every $u \in V$ $\quad 1 \leq i < j \leq k$

$$(\neg x_u^i \vee \neg x_u^j)$$

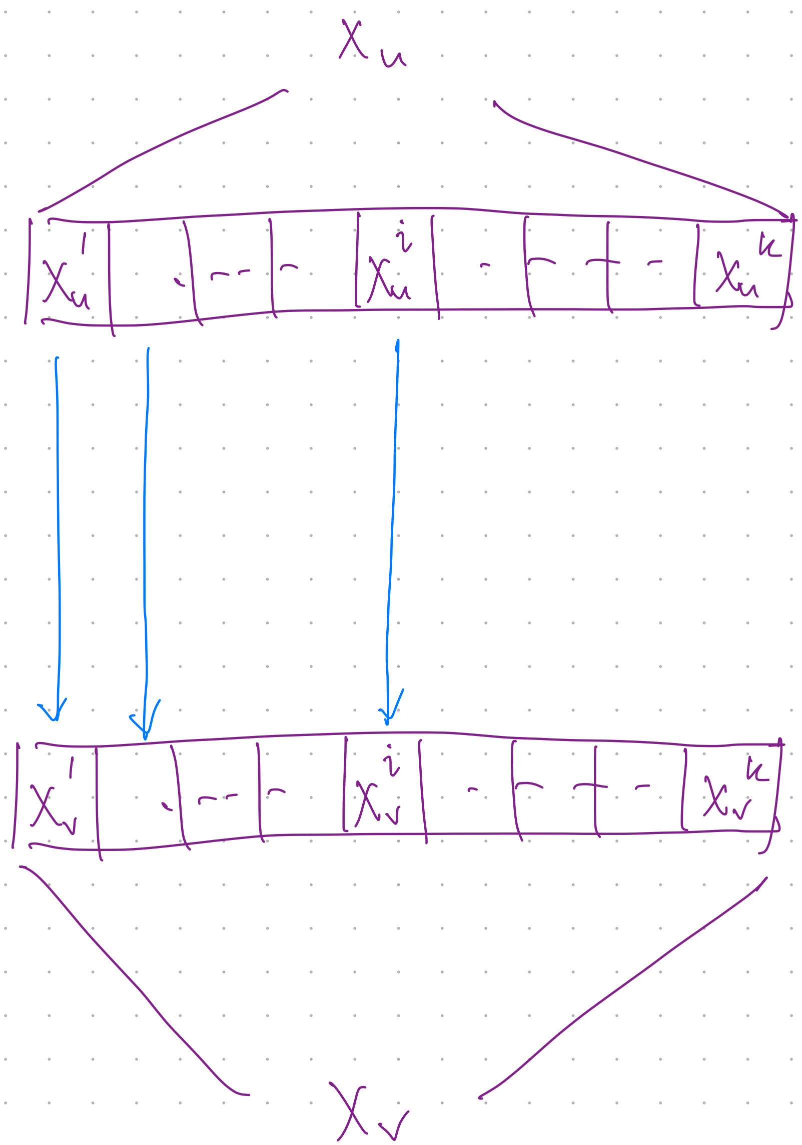
u cannot be i^{th} and j^{th} vertex of S

$x_u^i = T \iff u$ is the i^{th} vertex in S

Exactly 1 vertex must be the i^{th} vertex



$x_u^i = T \iff u$ is the i^{th} vertex in S

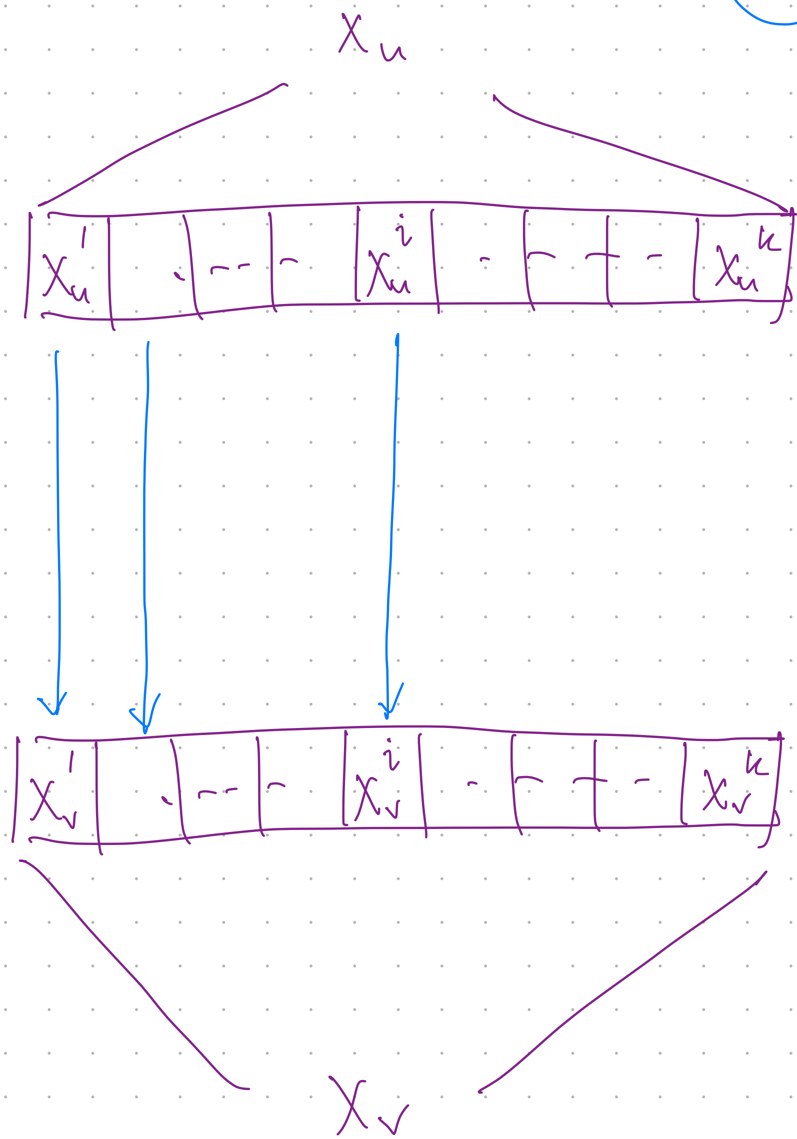


Exactly 1 vertex must be the i^{th} vertex

At least 1.

$$\left(x_u^i \vee x_v^i \vee x_w^i \vee \cdots_{\forall v \in V}\right)$$

$x_u^i = T \iff u$ is the i^{th} vertex in S



2 Exactly 1 vertex must be the i^{th} vertex

At least 1.

(2a) $\left(x_u^i \vee x_v^i \vee x_w^i \vee \dots_{\forall v \in V}\right)$

At most 1

(2b) $\forall u, v \in V$

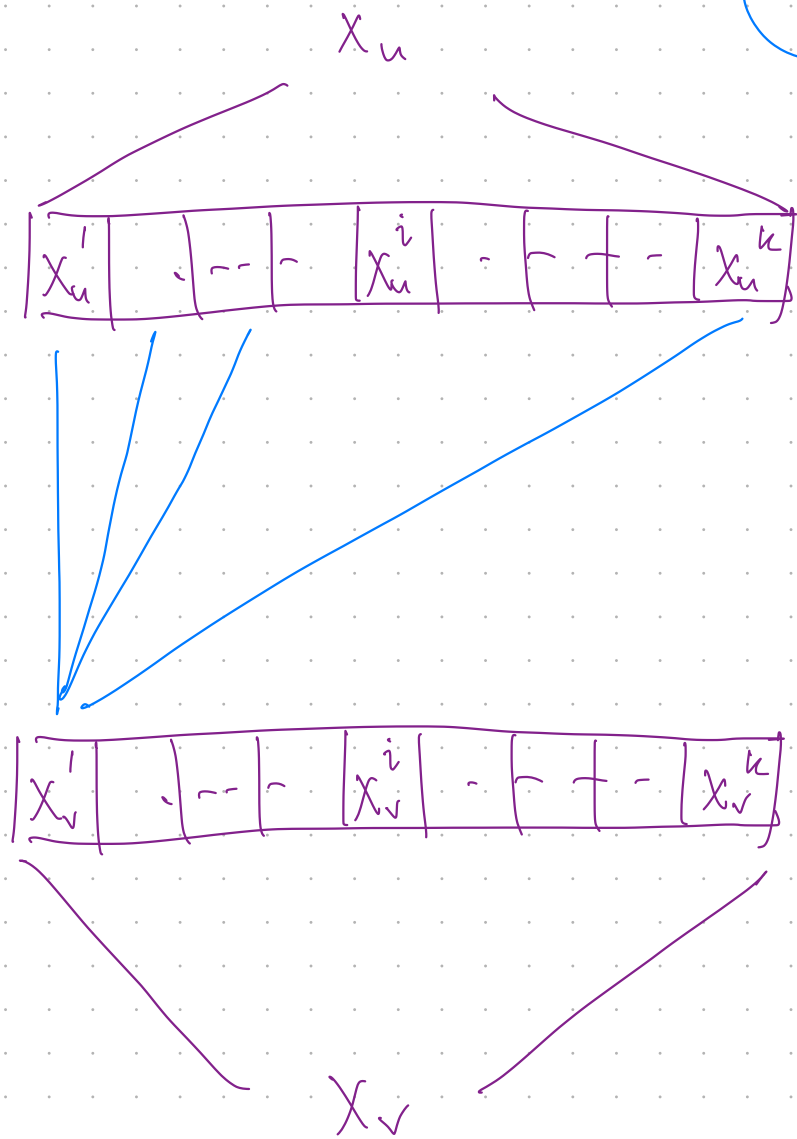
$$\left(\neg x_u^i \vee \neg x_v^i\right)$$

$x_u^i = T \iff u$ is the i^{th} vertex in S

3

IND SET constraints

No vertices in S share an edge



$\forall u, v \in E$

$\forall i, j \in \{1, \dots k\}$

$$(\neg x_u^i \vee \neg x_v^j)$$

$G = (V, E)$, k

$\downarrow$

$$\varphi_{G,k} = \bigwedge_{u \in V} (\neg x_u^1 \vee \neg x_u^2) \wedge (\neg x_u^1 \vee \neg x_u^3) \wedge \dots \wedge (\neg x_u^{k-1} \vee \neg x_u^k) \quad (1)$$

$$\wedge\ (x_u^1 \vee x_v^1 \vee \dots \vee x_w^1) \wedge (x_u^2 \vee x_v^2 \vee \dots \vee x_w^2) \wedge \dots \wedge (x_u^k \vee x_v^k \vee \dots \vee x_w^k) \quad (2a)$$

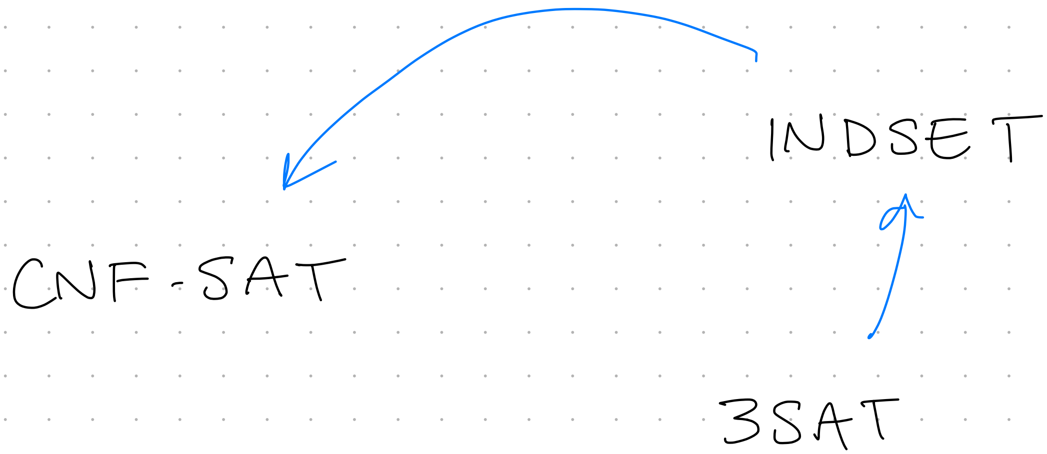
$$\wedge\ (\neg x_u^1 \vee \neg x_v^1) \wedge \dots \wedge (\neg x_v^1 \vee \neg x_w^1) \wedge \dots \wedge (\neg x_u^k \vee \neg x_v^k) \wedge \dots \wedge (\neg x_v^k \vee \neg x_w^k) \quad (2b)$$

$$\bigwedge_{(u,v) \in E} (\neg x_u^1 \vee \neg x_v^2) \wedge (\neg x_u^1 \vee \neg x_v^3) \wedge \dots \wedge (\neg x_u^{k-1} \vee \neg x_v^k) \quad (3)$$

polynomial-sized CNF

INDSET $\leq_p$ CNF-SAT.

Previously we saw 3SAT $\leq_p$ INDSET



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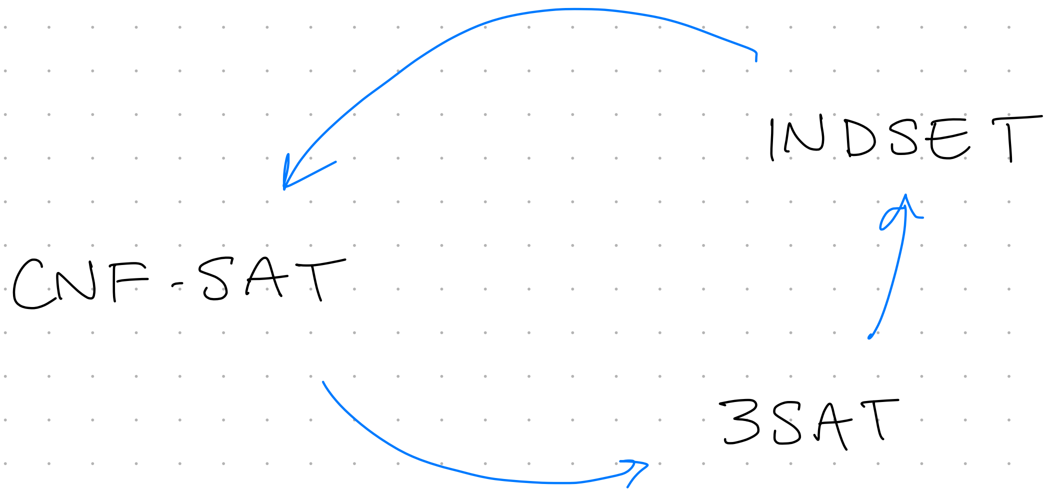


NP-Complete problems form an equivalence class (under poly-time Karp reductions)

INDSET $\leq_p$ CNF-SAT.

Previously we saw 3SAT $\leq_p$ INDSET

INDSET

CNF-SAT

3SAT

SUBSET-SUM

MAX CUT

CLIQUE

VERTEX-COVER

TSP

CIRCUIT SAT

HAMCYCLE

PROTEIN-FOLDING

INTEGER PROGRAMMING

If **any** NP-Complete problem has a poly-time algorithm, they **all** do!

(i.e. P = NP)

NP-Complete problems form an equivalence class

(under poly-time Karp reductions)

CIRCUIT - SAT $\leq_p$ 3SAT

Given: Logical Circuit $C : \{0,1\}^n \longrightarrow \{0,1\}$

Question:

Does there exist $x \in \{0,1\}^n$

s.t. $C(x) = 1$?

CIRCUIT - SAT $\leq_p$ 3SAT

Given: Logical Circuit $C : \{0,1\}^n \longrightarrow \{0,1\}$

Question.

Does there exist $x \in \{0,1\}^n$

s.t. $C(x) = 1$?

Note. CIRCUIT SAT $\leq_p$ 3SAT is a key step in proof of Cook-Levin Theorem!

Intuition. Logical Circuits can implement any algorithm.

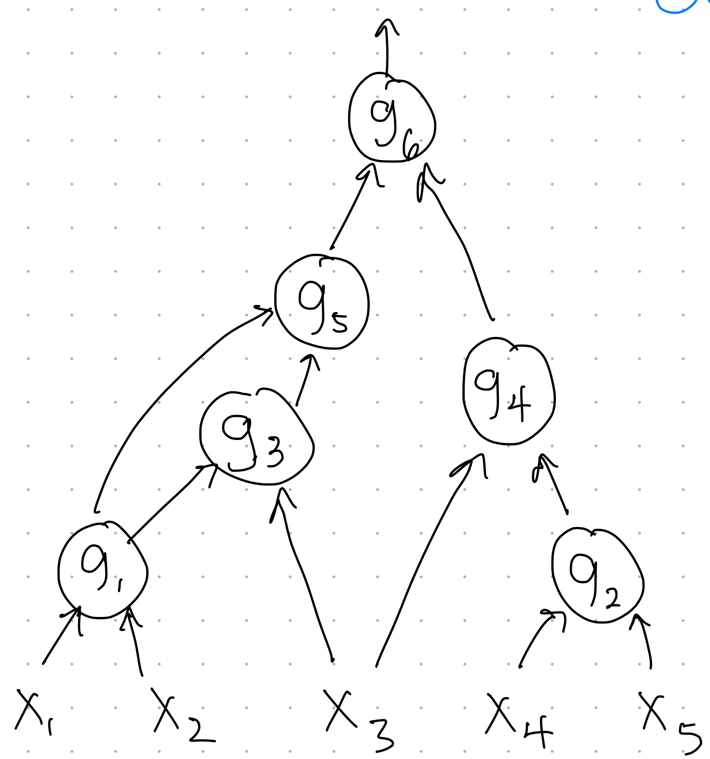
Circuits.

* Represented as a DAG
* Vertices ≡ "Gates"
* Edges ≡ "Wires"

// Each gate computes a boolean fn. on 2-variables

→ n total input wires to circuit

→ Output determined by evaluating each gate from bottom to top.



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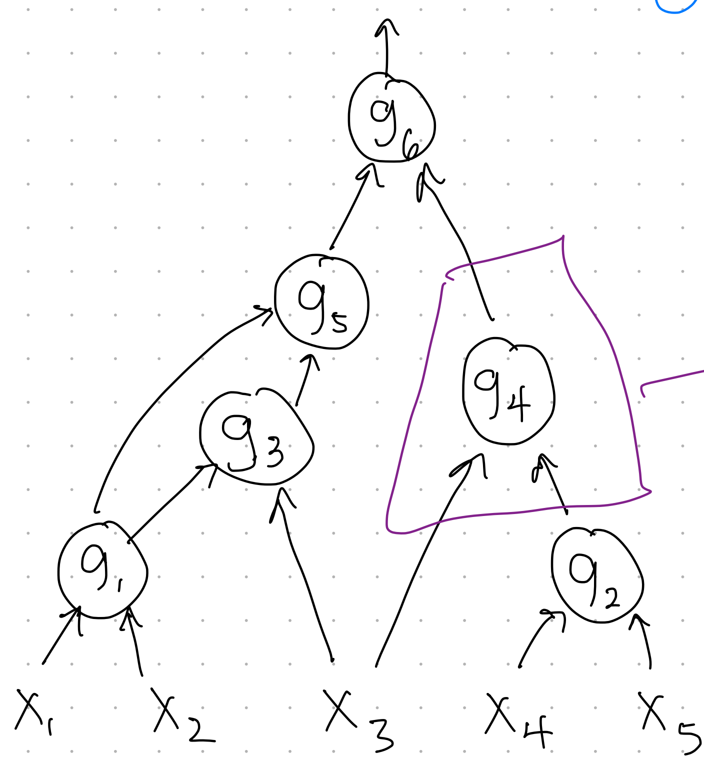
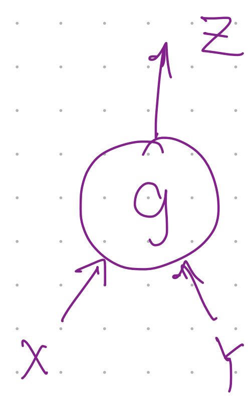


Reduction idea

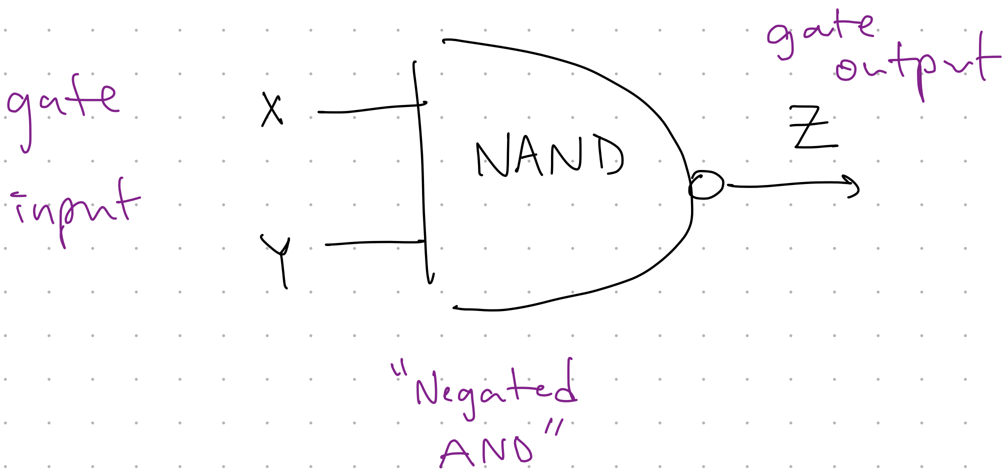
For each gate



Check correctness

$$z \longleftrightarrow g(x, y)$$

Verify each gate

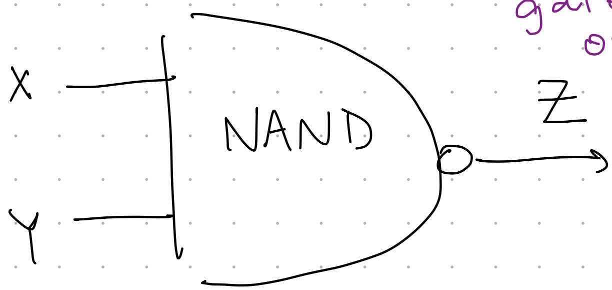


X	Y	Z = NAND(X, Y)
0	0	1
1	0	1
0	1	1
1	1	0

Note: NAND is Complete

Verify each gate

gate input



gate output

"Negated AND"

X	Y	Z = NAND(X, Y)
0	0	1
1	0	1
0	1	1
1	1	0

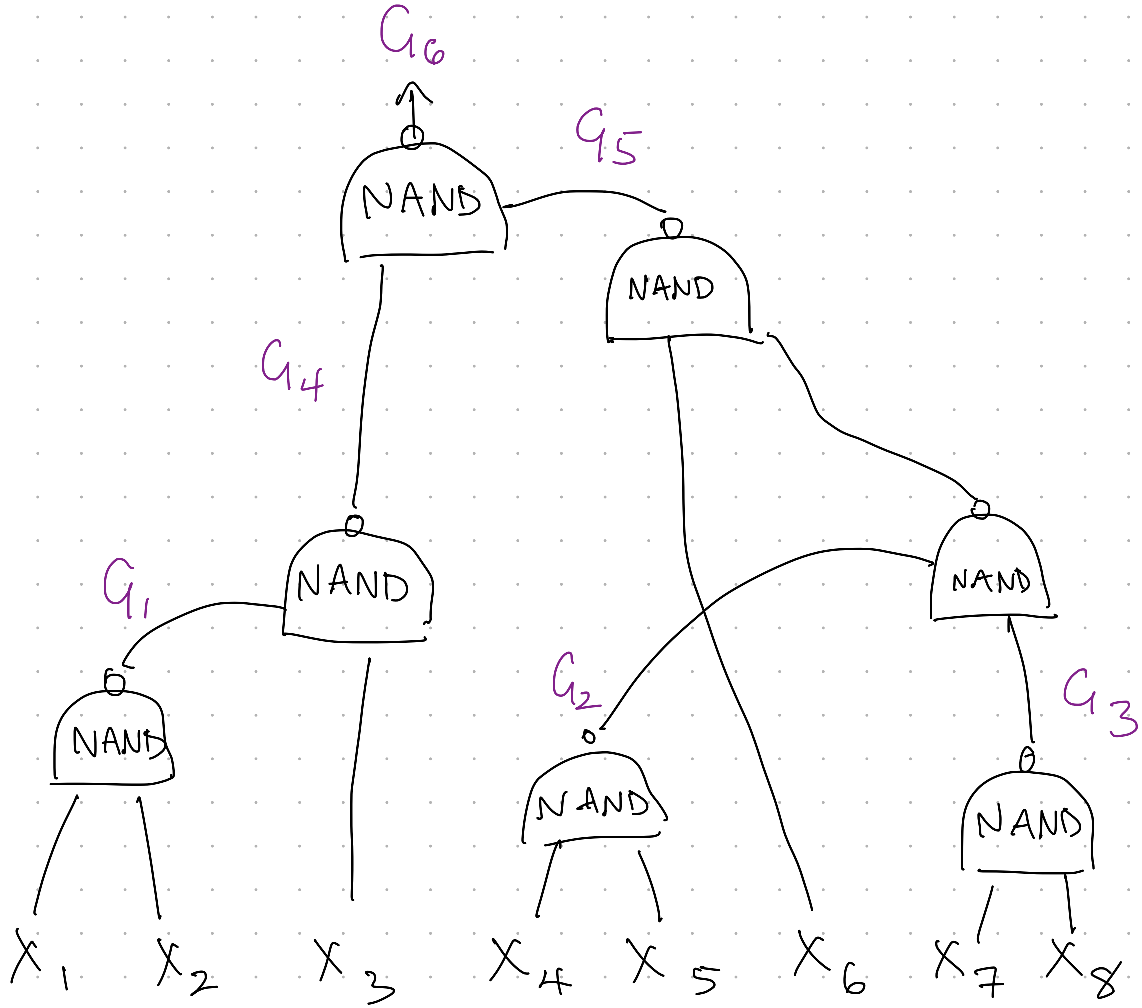
Note: NAND is Complete

GATE GADGET. $\longrightarrow$ Set of clause satisfied by any X, Y, Z s.t. Z = NAND(X, Y)

$$Z \leftrightarrow \neg(X \wedge Y)$$

$$\Leftrightarrow \quad \underbrace{Z \to \neg(X \wedge Y)}_{(\neg Z \vee \neg X \vee \neg Y)} \quad \wedge \quad \underbrace{\neg Z \to (X \wedge Y)}_{(Z \vee X) \wedge (Z \vee Y)}$$

$$(\neg Z \vee \neg X \vee \neg Y) \wedge (Z \vee X) \wedge (Z \vee Y)$$



poly-sized CNF satisfiable iff $\exists\, x \in \{0,1\}^8$ s.t. $C(x) = 1$.

$$\varphi_C = G_6 \wedge \Big[(\neg G_1 \vee \neg x_1 \vee \neg x_2) \wedge (G_1 \vee x_1) \wedge (G_1 \vee x_2) \quad \text{(gate 1 gadget)}$$

$$\wedge\ (\neg G_2 \vee \neg x_4 \vee \neg x_5) \wedge (G_2 \vee x_4) \wedge (G_2 \vee x_5) \quad \text{(gate 2 gadget)}$$

$$\vdots$$

$$\wedge\ (\neg G_6 \vee \neg G_4 \vee \neg G_5) \wedge (G_6 \vee G_4) \wedge (G_6 \vee G_5) \Big] \quad \text{(gate 6 gadget)}$$