22 Mar Vertex Cover and Hamiltonian Cycle

Announcements about PSet 7.

① Q1 early turn-in option.

Turn in $\leq$ Tues 4pm

$\Rightarrow$ graded before end of Sp Brk.

② Q2 has a 4820 and a 5820 version.

This will be last homework before Prelim 2.

Prelim 2 not cumulative: covers Ch. 5, 7, 8.

(div-conq, flow, NP-Complete problems)

DEFINITIONS.

1. **NP** is the class of decision problems for which there exists a polynomial-time algorithm to verify a "yes" answer, given a suitable hint.

If Π is a problem, we say $\Pi \in NP$ if there's an algorithm V with two inputs.

$V(x, y)$ — x denotes an instance of Π

y denotes a "hint" or "candidate solution"

(a) V runs in poly time in $\underbrace{|x|}_{\text{length of } x} + \underbrace{|y|}_{\text{length of } y}$.

(b) $\Pi(x) = \text{yes}$ if and only if

$\exists y$ s.t. $|y| \leq O(\text{poly}(|x|))$ and $V(x,y) = \text{yes}$.

E.g. 3SAT.

x = specification of variables & clauses

y = specification of truth assignment

$V(x,y)$ = algorithm that checks y satisfies every clause specified in x.

$P \subseteq NP$ because $V(x,y)$ can ignore y and solve x in poly time.

$P \stackrel{?}{=} NP$: most believe $P \neq NP$.

2. Π is NP-Hard if any of the following equivalent statements hold.

Cook-Levin Theorem (CS 4810/4814)

a. Every problem $\Pi' \in NP$ satisfies $\Pi' \leq_p \Pi$.

b. There exists an NP-Hard Π' s.t. $\Pi' \leq_p \Pi$.

c. $3SAT \leq_p \Pi$.

3. Π is NP-Complete if $\Pi \in NP$ and is NP-hard.

You can verify a solution of Π in poly-time but can't find a solution unless $P = NP$.

"Prove some problem ABC is NP-Complete."

easy/short

careful thought

1. Present a polytime verifier. (Show $ABC \in NP$.)
2. Present a reduction from some other NP-hard problem, XYZ, to ABC.
3. Show reduction runs in poly time.
4. Reduction, applied to "yes" instance of XYZ, yields "yes" instance of ABC.

5 Same as step 4, for "no" instances.

"gadgets work as intended"

"gadgets cannot be successfully used in unintended ways"!

often accomplish step 5 by proving its contrapositive

HAMILTONIAN CYCLE: input is a directed graph. Question: does the graph contain a simple cycle that visits all vertices?

(cycle with no repeated vertices)

1 Belongs to NP because a verifier given the graph, and a list of vertices in the cycle, checks that each vertex is on the cycle once and only once, and each edge of the cycle belongs to the graph.

2 Reduce from something NP-Hard

Textbook: 3SAT $\leq_p$ HAM CYCLE.

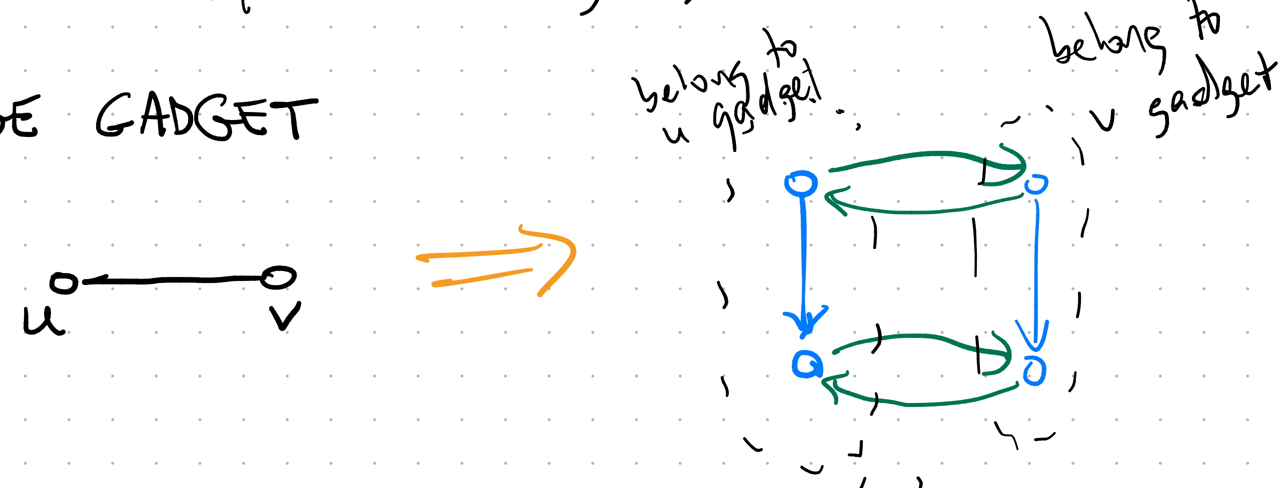
(Read it! It's instructive.)

Today: VERTEX COVER $\leq_p$ HAM CYCLE.

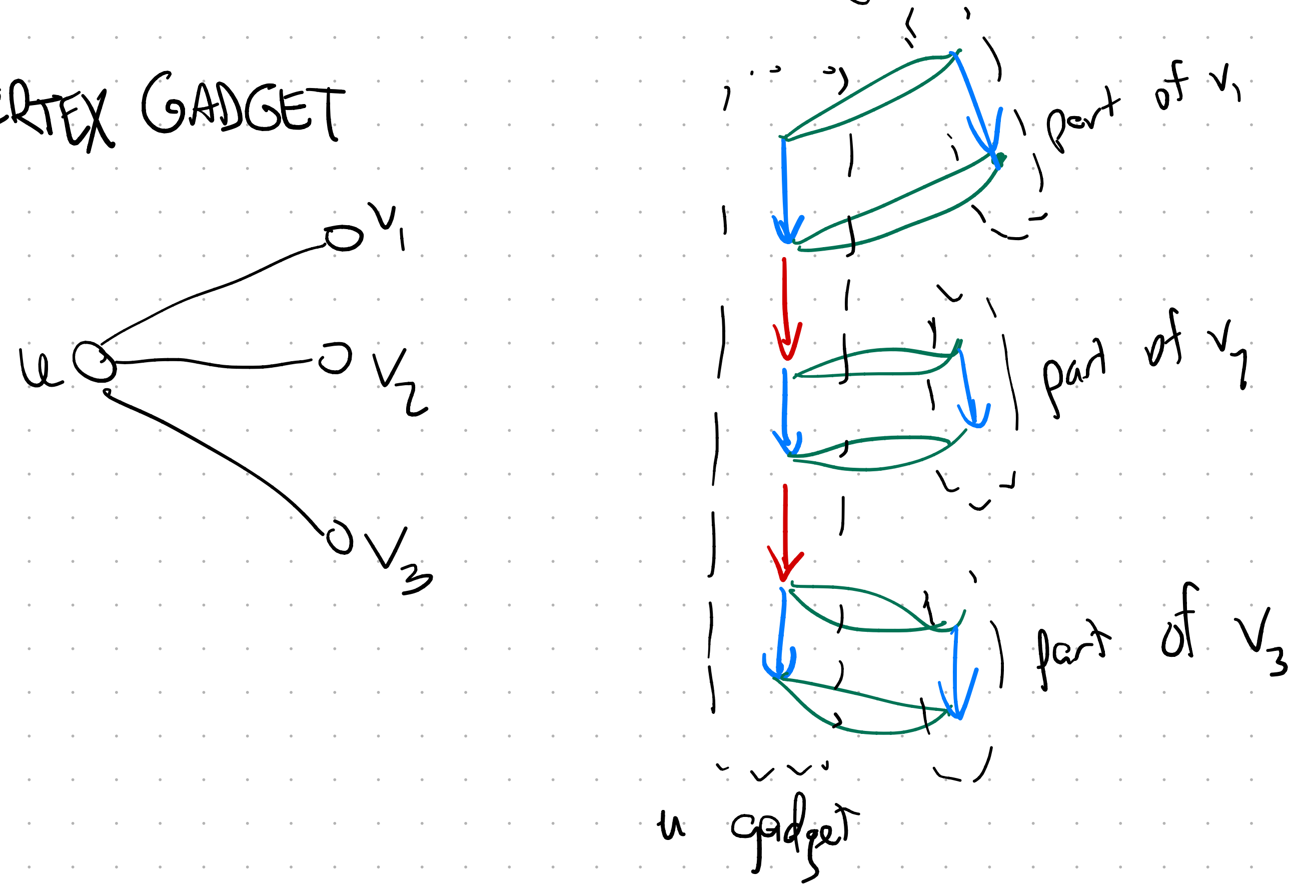
Given undirected G, $k \in \mathbb{N}$.
Can we find $\leq k$ vertices in G such that each edge has at least one endpoint among the k vertices?

Will represent G as a HAM CYCLE input with 3 types of gadgets.

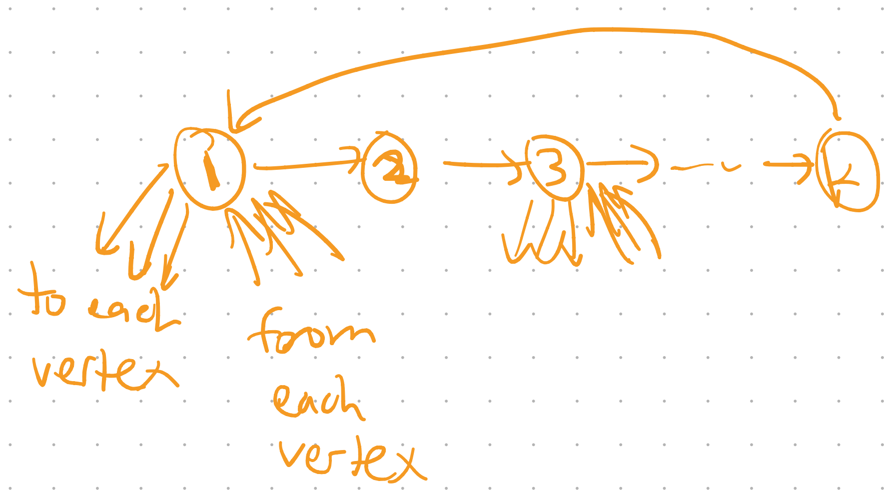
EDGE GADGET

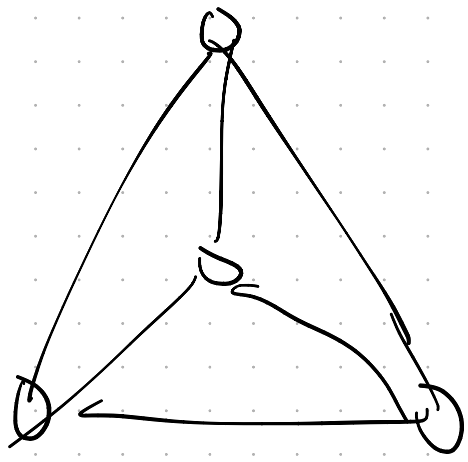


VERTEX GADGET



COUNTER GADGET





$k = 3$