11 March 2024   Max Flow - Min Cut

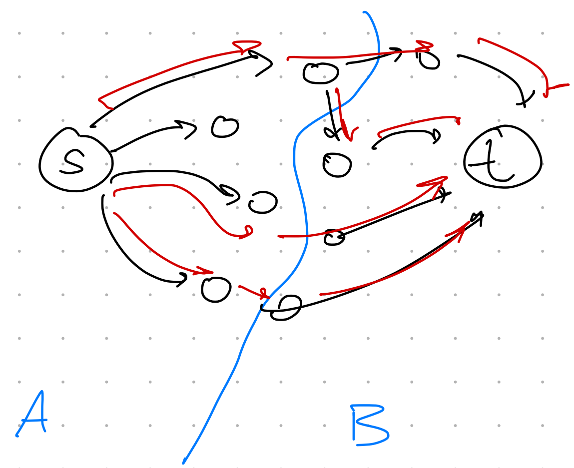
Plan.

* Finish proof of Max Flow - Min Cut Theorem
* Announcements
* Max Bipartite Matching.

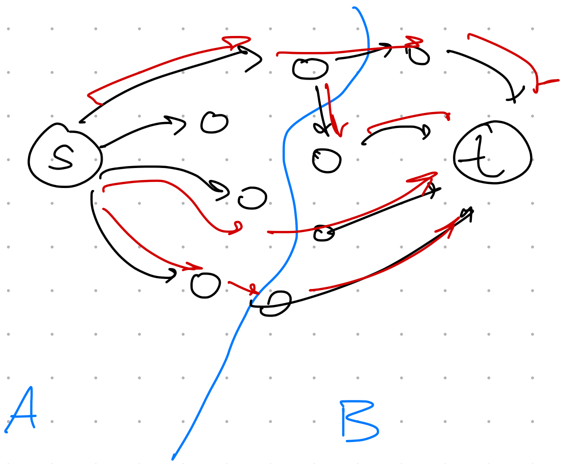
Recall. Relationship between st-flows and st-cuts

st-cut: $A \subseteq V$ s.t. $s \in A$

$B = V \setminus A$ s.t. $t \in B$



Recall. Relationship between st-flows and st-cuts



st-cut: $A \subseteq V$ s.t. $s \in A$

$B = V \setminus A$ s.t. $t \in B$

Claim. Fix a flow f and any st-cut A, B.

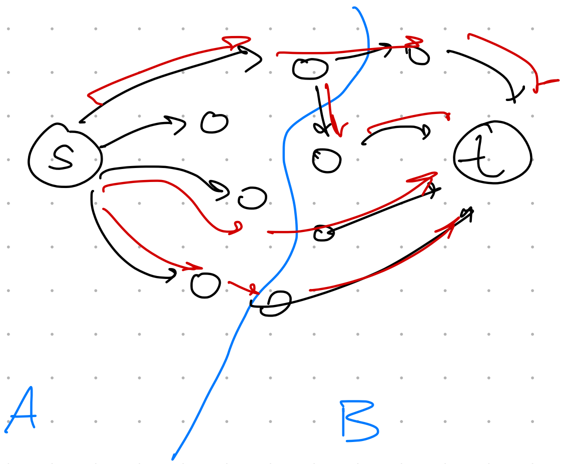
$$\mathrm{val}(f) = f^{out}(A) - f^{in}(A)$$

$$= \sum_{\substack{u \in A \\ v \in B}} \left(f_{uv} - f_{vu} \right)$$

Recall. Relationship between st-flows and st-cuts

st-cut: $A \subseteq V$ s.t. $s \in A$

$B = V \setminus A$ s.t. $t \in B$



Claim. Fix a flow f and any st-cut A, B.

$$\mathrm{val}(f) = f^{out}(A) - f^{in}(A)$$

$$= \sum_{\substack{u \in A \\ v \in B}} \left(f_{uv} - f_{vu} \right)$$

$$\leq \sum_{\substack{u \in A \\ v \in B}} c_{uv} = \mathrm{cap}(A, B)$$

Lemma. For any flow f and st-cut (A,B)

$$val(f) \le cap(A,B).$$

Corollary

$$\max_{f^*} val(f^*) \le \min_{A^*, B^*} cap(A^*, B^*)$$

Max Flow - Min Cut Theorem.

max flow $\leq$ min cut

Max Flow - Min Cut Theorem.

max flow = min cut

Max Flow - Min Cut Theorem.

$$\text{max flow} = \text{min cut}$$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.

That is, $val(f_{FF}) \geq \min_{A^*,B^*} cap(A^*,B^*)$

Max Flow - Min Cut Theorem.

$$\text{max flow} = \text{min cut}$$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.

That is, $val(f_{FF}) \geq \min_{A^*,B^*} cap(A^*,B^*)$

Corollary. Ford-Fulkerson returns a max flow.

(Establishes correctness of FF)

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $\text{val}(f_{FF}) = \text{cap}(A,B)$.

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.

FF Termination Condition. No path from $s \to t$ in G_f.

Define cut A,B $\quad A = \{ u \in V : u$ is reachable from s in $G_f \}$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $\text{val}(f_{FF}) = \text{cap}(A,B)$.

FF Termination Condition. No path from $s \to t$ in G_f.

Define cut A,B $\quad A = \{ u \in V : u \text{ is reachable from } s \text{ in } G_f \}$

Note. By termination, $t \notin A \Rightarrow A,B$ is a st-cut.

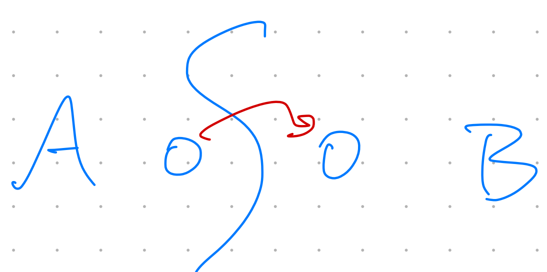
Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.

FF Termination Condition. No path from $s \to t$ in G_f.

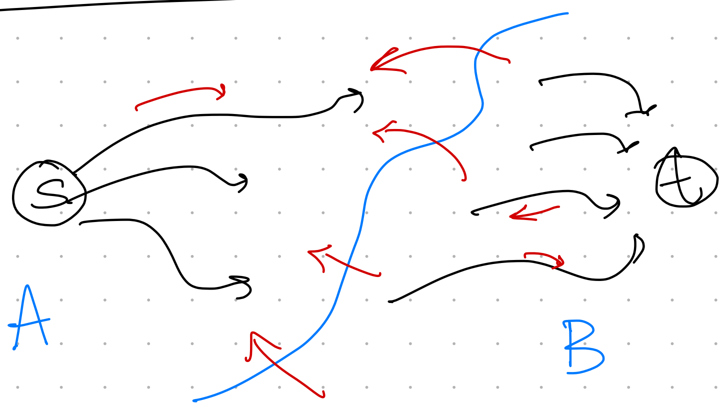
Define cut A,B $\quad A = \{u \in V : u$ is reachable from s in $G_f\}$

$\boxed{\text{Key Observation}}$ By defn. of A, No residual edges go from $A \longrightarrow B$

$(*)$ $\forall u \in A, \forall v \in B \quad uv \notin E(G_f)$

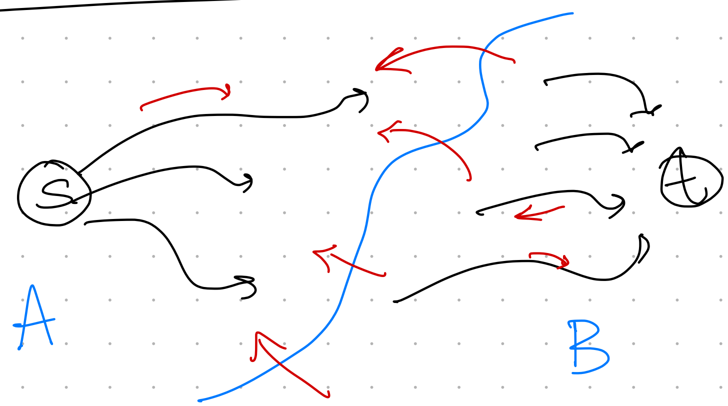


<u>Lemma.</u> When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.



$$\boxed{(*)\quad \forall u \in A,\ \forall v \in B,\quad uv \notin E(G_f)}$$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.



$$(*) \quad \forall u \in A, \forall v \in B, \quad uv \notin E(G_f)$$

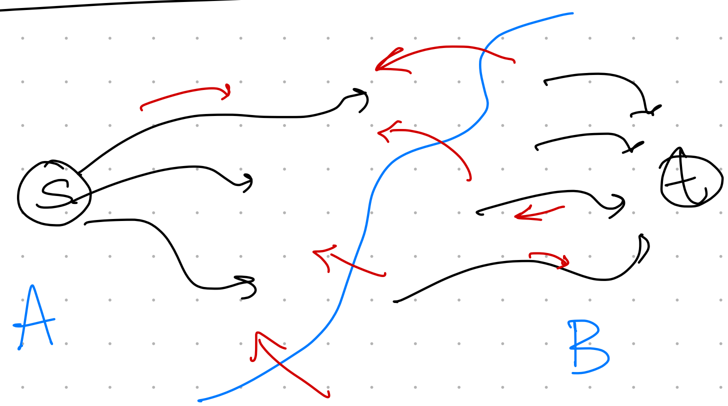
①

$\forall u \in A \ \forall v \in B$

$uv \in E(G) \Rightarrow f_{uv} = c_{uv}$

if edge not at capacity
then $uv \in G_f$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $val(f_{FF}) = cap(A,B)$.



$$\boxed{(*) \quad \forall u \in A, \forall v \in B, \quad uv \notin E(G_f)}$$

①

$\forall u \in A \ \forall v \in B$

$uv \in E(G) \Rightarrow f_{uv} = c_{uv}$

if edge not at capacity then $uv \in G_f$

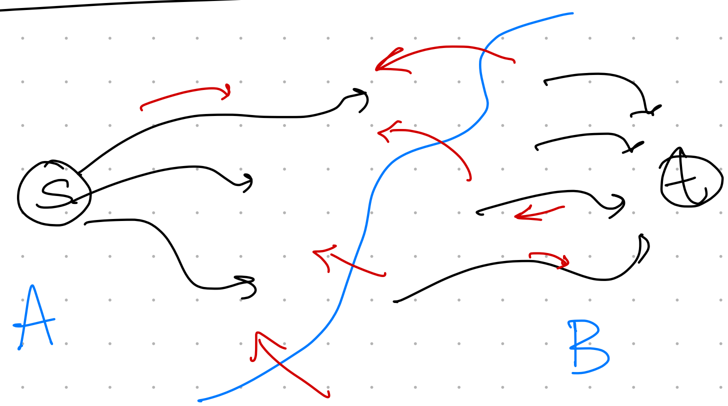
②

$\forall u \in A \ \forall v \in B$

$vu \in E(G) \Rightarrow f_{vu} = 0$

$f_{vu} > 0$ introduces a residual edge $A \to B$.

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $\mathrm{val}(f_{FF}) = \mathrm{cap}(A,B)$.



$\forall u \in A \ \forall v \in B$

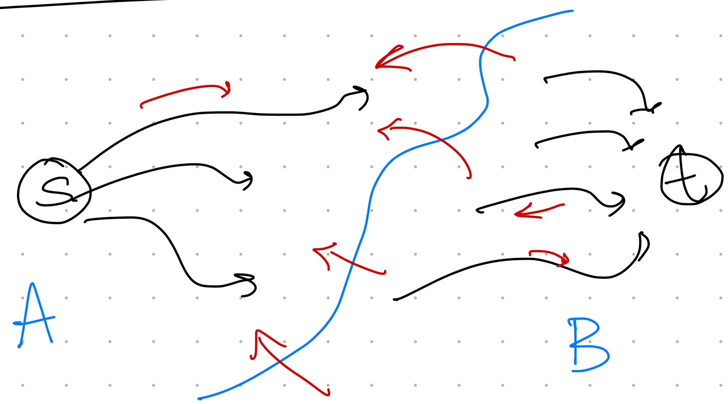
① $uv \in E(G) \Rightarrow f_{uv} = c_{uv}$

② $vu \in E(G) \Rightarrow f_{vu} = 0$

$$\mathrm{val}(f) = f^{out}(A) - f^{in}(A)$$

$$= \underbrace{\sum_{\substack{u \in A \\ v \in B}} f_{uv}}_{①} - \underbrace{\sum_{\substack{u \in A \\ v \in B}} f_{vu}}_{②}$$

Lemma. When Ford-Fulkerson terminates with flow f_{FF} there exists a cut (A,B) s.t. $\text{val}(f_{FF}) = \text{cap}(A,B)$.



$\forall u \in A \;\; \forall v \in B$

① $uv \in E(G) \Rightarrow f_{uv} = c_{uv}$

② $vu \in E(G) \Rightarrow f_{vu} = 0$

$$\text{val}(f) = f^{out}(A) - f^{in}(A)$$

$$= \underbrace{\sum_{\substack{u\in A\\ v\in B}} f_{uv}}_{①} - \underbrace{\sum_{\substack{u\in A\\ v\in B}} f_{vu}}_{②}$$

$$= \sum_{\substack{u\in A\\ u\in B}} c_{uv} - 0 = \text{cap}(A,B)$$

Corollary. Ford-Fulkerson returns a max flow f^*.

Corollary. Ford-Fulkerson returns a max flow f^*.

Corollary. Let (A^*, B^*) be the cut defined by

$$A^* = \{ u \in V : u \text{ reachable from } s \text{ in } G_{f^*} \}$$

(A^*, B^*) is a min cut.

<u>Corollary</u>. Ford-Fulkerson returns a max flow f^*.

<u>Corollary</u>. Let (A^*, B^*) be the cut defined by

$$A^* = \{ u \in V : u \text{ reachable from } s \text{ in } G_{f^*} \}$$

(A^*, B^*) is a min cut.

<u>Corollary</u>. There exists an $O(m \cdot V)$ algorithm to compute the min st-cut.

Different heuristics for choosing an augmenting path have different performance

Different heuristics for choosing an augmenting path have different performance

Examples.

* max "bottleneck" path

$$\max_{p \in s \rightsquigarrow t} \; \min_{e \in p} \; c_f(e)$$

poly$(m, \log |U|)$

"weakly polynomial"

Different heuristics for choosing an augmenting path have different performance

Examples.

* max "bottleneck" path

$$\max_{p \in s \rightsquigarrow t} \min_{e \in p} c_f(e)$$

poly$(m, \log |U|)$

"weakly polynomial"

* shortest hops

$$\min_{p \in s \rightsquigarrow t} |p|$$

poly(n, m)

Edmonds-Karp: $O(m^2 n)$ "strongly polynomial"

Different heuristics for choosing an augmenting path have different performance

Examples.

* max "bottleneck" path

$$\max_{p \in s \rightsquigarrow t} \min_{e \in p} c_f(e)$$

poly$(m, \log |U|)$

"weakly polynomial"

* shortest hops

$$\min_{p \in s \rightsquigarrow t} |p|$$

poly(n, m)

Edmonds-Karp: $O(m^2 n)$ "strongly polynomial"

Current Max-flow RT: $O(m^{1+\varepsilon})$ for all $\varepsilon > 0$

↳ 2023

"near-linear"

Maximum Matching.

Given: Undirected Graph $G = (V, E)$

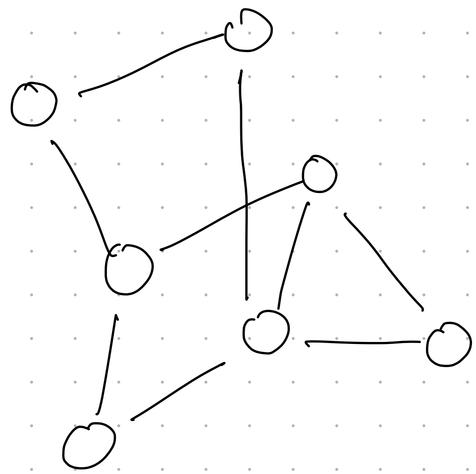
Find: maximum cardinality matching $M \subseteq E$.

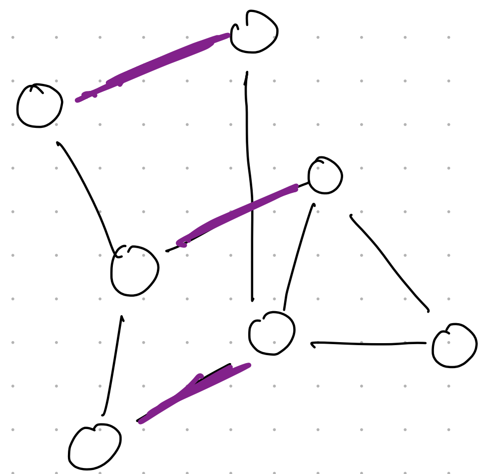
A matching is a subset of the edges M where no two edges in M share an endpoint.

Maximum Matching.

Given : Undirected Graph $G = (V, E)$

Find : maximum cardinality matching $M \subseteq E$.

$G =$ 

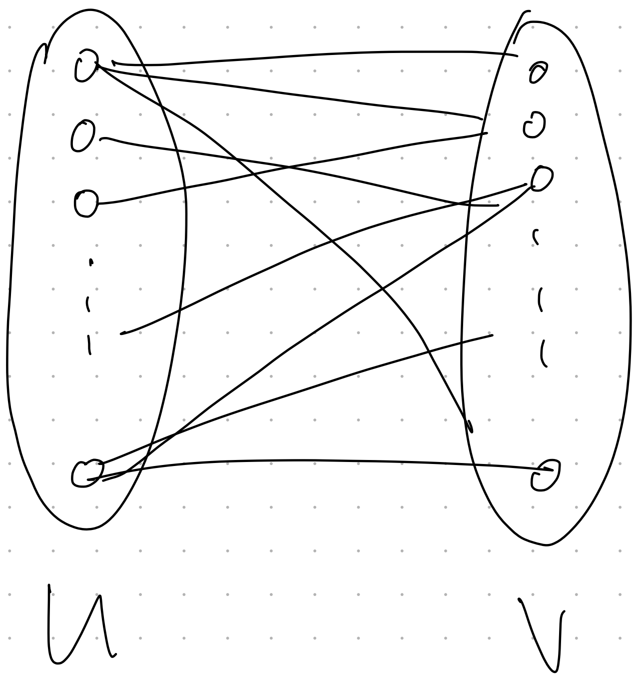
$M =$ 

A matching is a subset of the edges M where no two edges in M share an endpoint.

Maximum Bipartite Matching.

Given: Bipartite Graph $G = (U, V, E)$

Find: Max cardinality matching

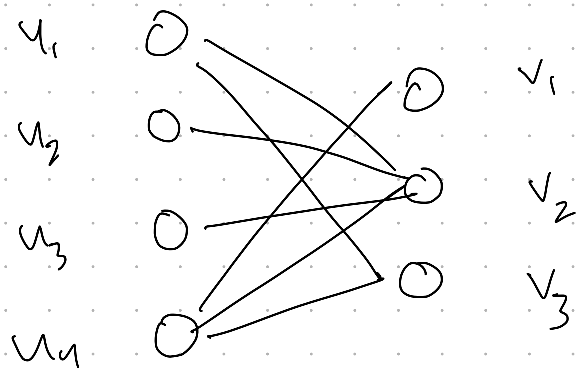


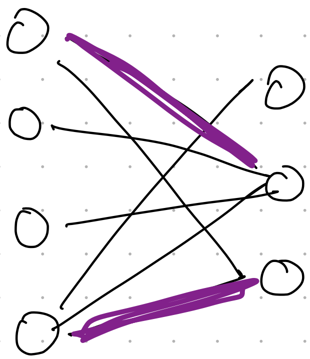
Bipartite Graph

vertices can be partitioned into U, V s.t.

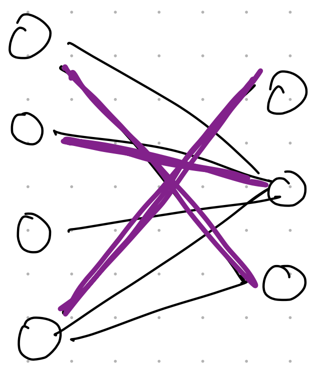
$\forall (u,v) \in E$

$u \in U$, $v \in V$



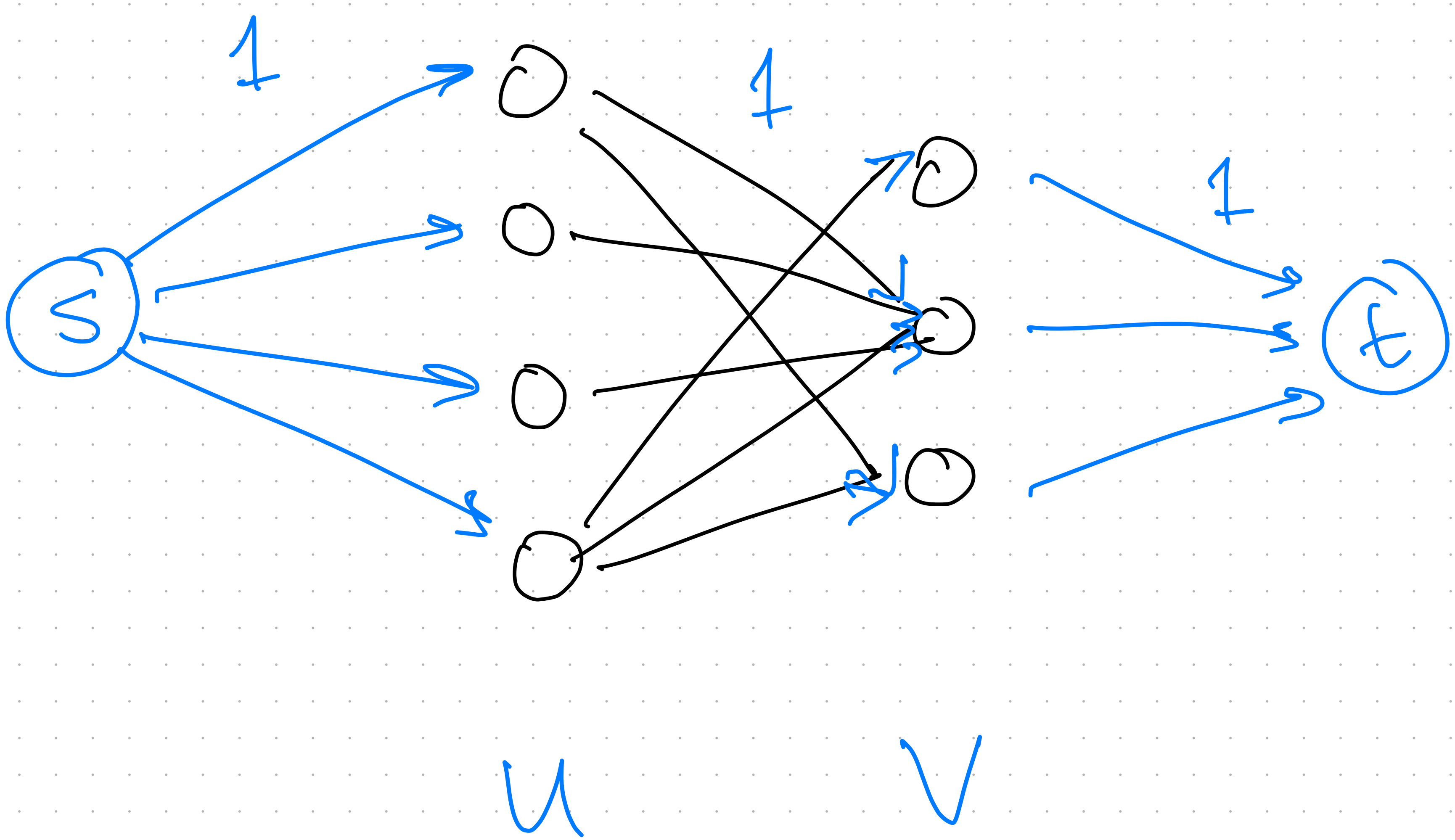


$$M = \{ (u_1, v_2), \ (u_4, v_3) \}$$

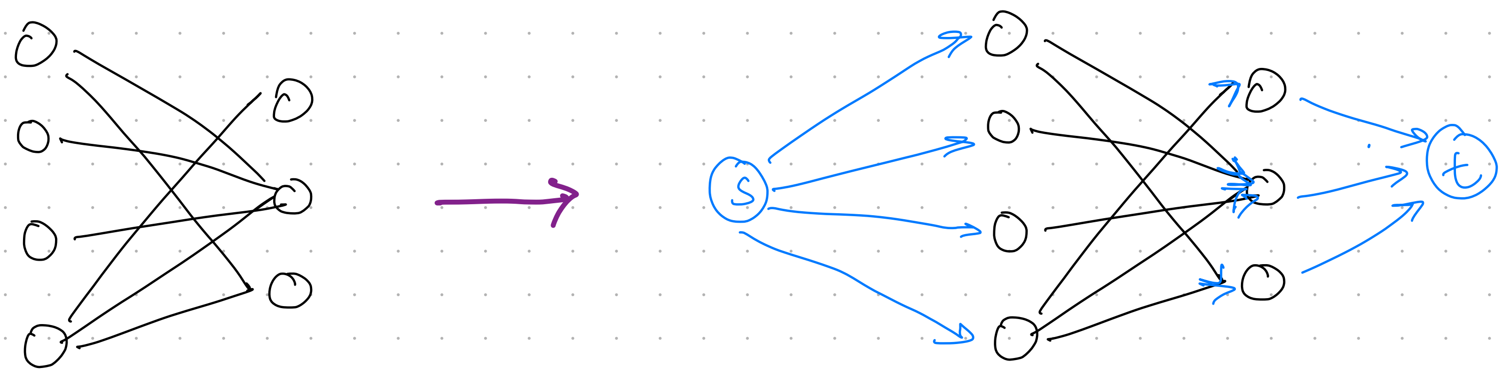


$$M' = \{ (u_1, v_3), (u_2, v_2), (u_3, v_1) \}$$

Algorithm for solving Max Bipartite Matching?



Reduction to Max Flow



Max Bipartite Match(U, V, E)

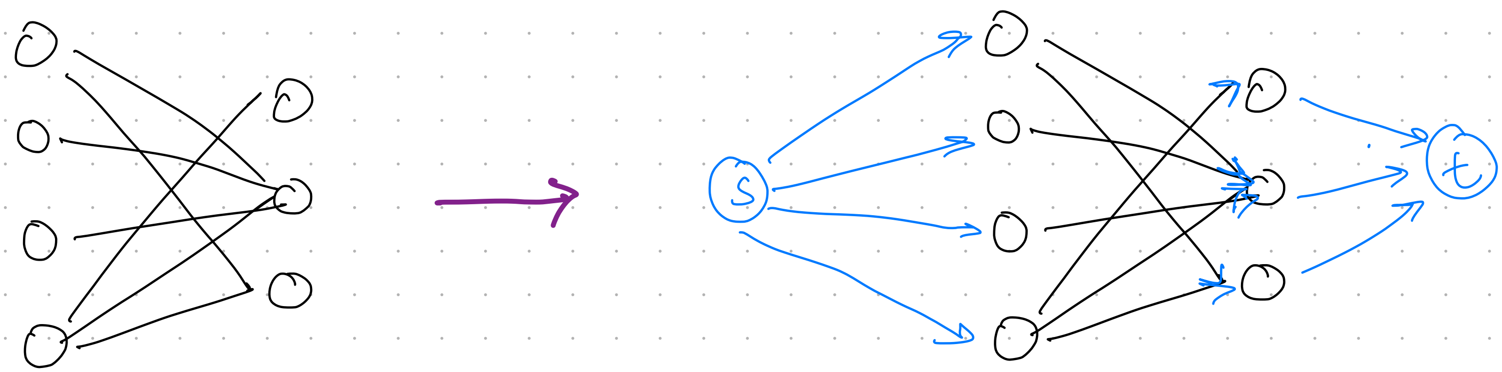
Construct directed graph G'

For all edges e in G'

Capacity $c_e = 1$

Return MaxFlow(G', s, t, c)

Reduction to Max Flow



Max Bipartite Match (U, V, E):

Construct directed graph G'

For all edges e in G'

Capacity $c_e = 1$

Return MaxFlow(G', s, t, c)

Vertices: $U \cup V \cup \{s, t\}$

Edges:

* for all $uv \in E$
  add $u \to v$
* for all $u \in U$
  add $s \to u$
* for all $v \in V$
  add $v \to t$

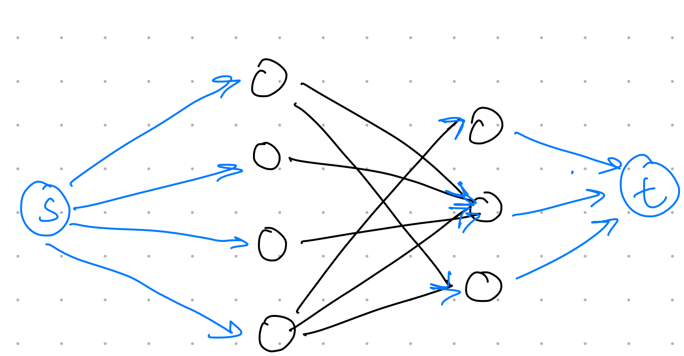
Max Bipartite Match (U, V, E).

Construct directed graph G'

For all edges e in G'

Capacity $c_e = 1$

Return MaxFlow(G', s, t, c)



Claim. The max cardinality of a matching in G equals the max flow in G'