29 January — Greedy Algorithms

Today's Plan

1. Interval Scheduling Problem
2. Intermission
   - ↳ Announcements
   - ↳ Reductions
3. Greedy Paradigm
   - ↳ Greedy Stays Ahead
   - ↳ Solution to Interval Scheduling

Classic Motivation:

* single central processor
* many job requests

Question: How do we schedule the jobs?

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* single central processor
* many job requests

Question: How do we schedule the jobs?

Details

- Each job has a proposed start time & finish time
- Processor can handle at most 1 job at a time
- Assumption: jobs have equal priority

Interval Scheduling Problem

Given: List of n jobs, specified by [start, finish] time

$$\langle [s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n] \rangle$$

Goal: Return a set of non-conflicting jobs of maximum cardinality

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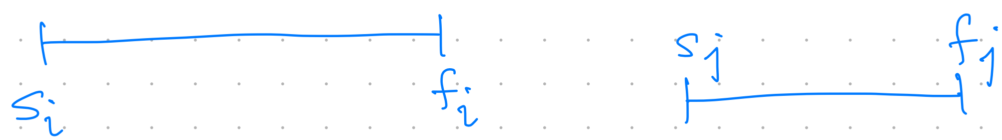

non-conflicting

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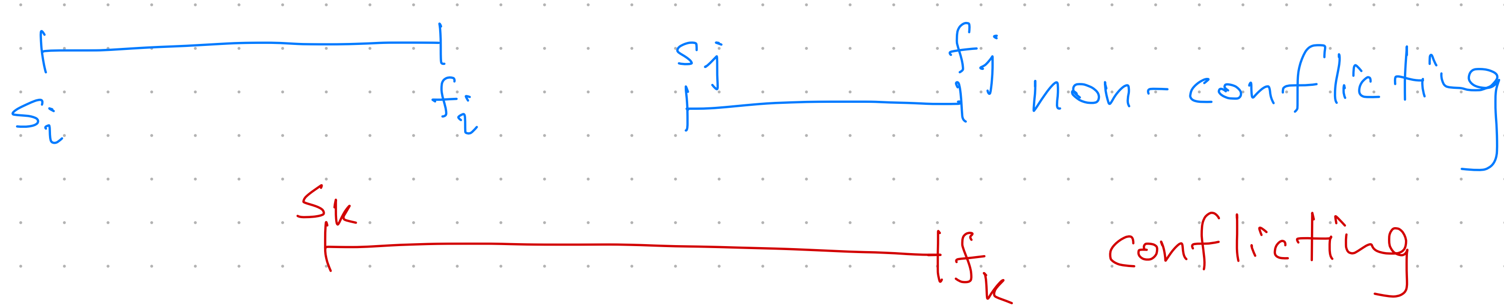

Announcements

* HW 1
  - Available on Canvas
  - Submission on Gradescope Open

* Significant Collaborators
  - Groups of ≤ 3 total
  - See Ed ~~#38~~ for partner search
    #26

* Enrollment in 4820 is capped
  - From CIS: No enrollment increase

A Note on Reductions

Problem P **reduces** to Problem Q if given an algorithm A_Q that solves Q, there exists an algorithm A_P (which makes calls to A_Q) that solves P.

4820 Philosophy: Look for Reductions!

* Useful for solving HW
* SPOILER: Reductions play essential role in showing certain problems are HARD.

Interval Scheduling Problem.

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Goal: Return a set of non-conflicting jobs of maximum cardinality

Example Applications?

i.e. What other problems reduce to Interval Scheduling?

Greedy Algorithms

Design Paradigm. Choose solution "greedily"

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Myopic → local decisions that "look good"

Irrevocable → once we make a decision, we never reconsider.

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Greedy Prototype.

1. Sort by "priority"

2. Iterate through elems in priority order
   ↳ Make decision about elem.

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Greedy Prototype.

1. Sort by "priority" ← Design of Greedy

   $O(n \log n)$

2. Iterate through elems in priority order

   ↳ Make decision about elem. $O(1)$

$$n \times O(1) = O(n).$$

RT Dominated by sorting!

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Warning. Challenging to analyze correctness

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Design Paradigm. Choose solution "greedily"

Myopic → local decisions that "look good"

Irrevocable → once we make a decision, we never reconsider.

Warning. Challenging to analyze correctness

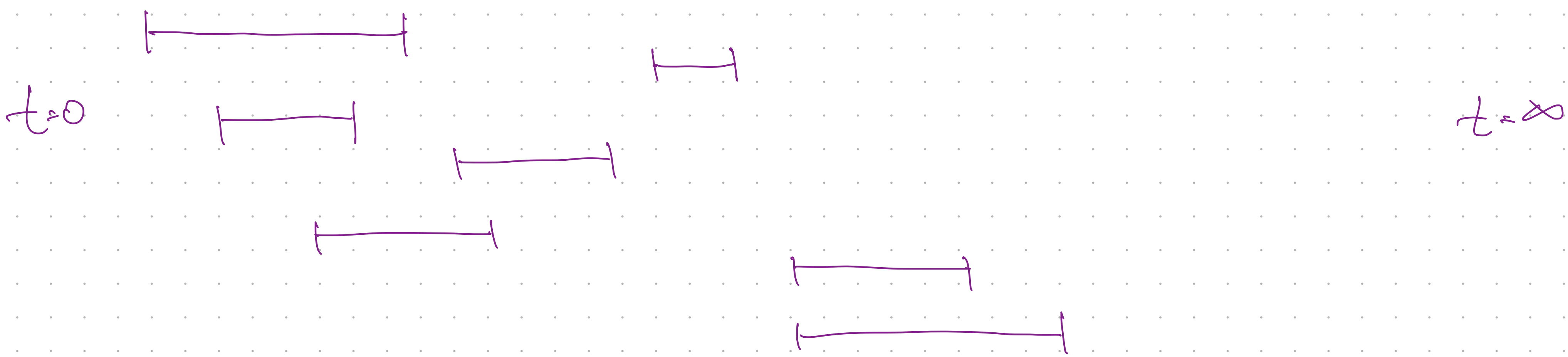
Often, Greedy approaches are incorrect.

Interval Scheduling Problem.

Given: List of n jobs, specified by [start, finish] time

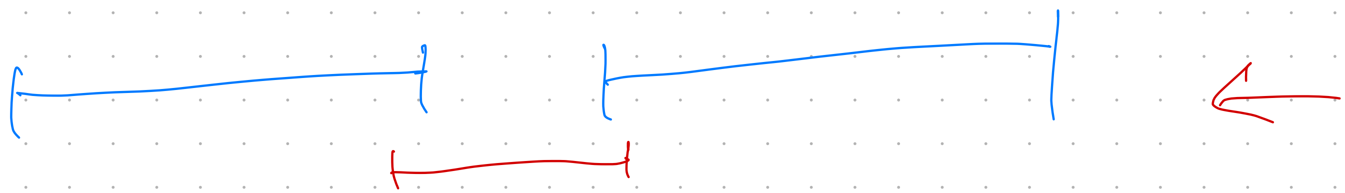
$$\langle [s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n] \rangle$$

Candidate "Priority" Functions:

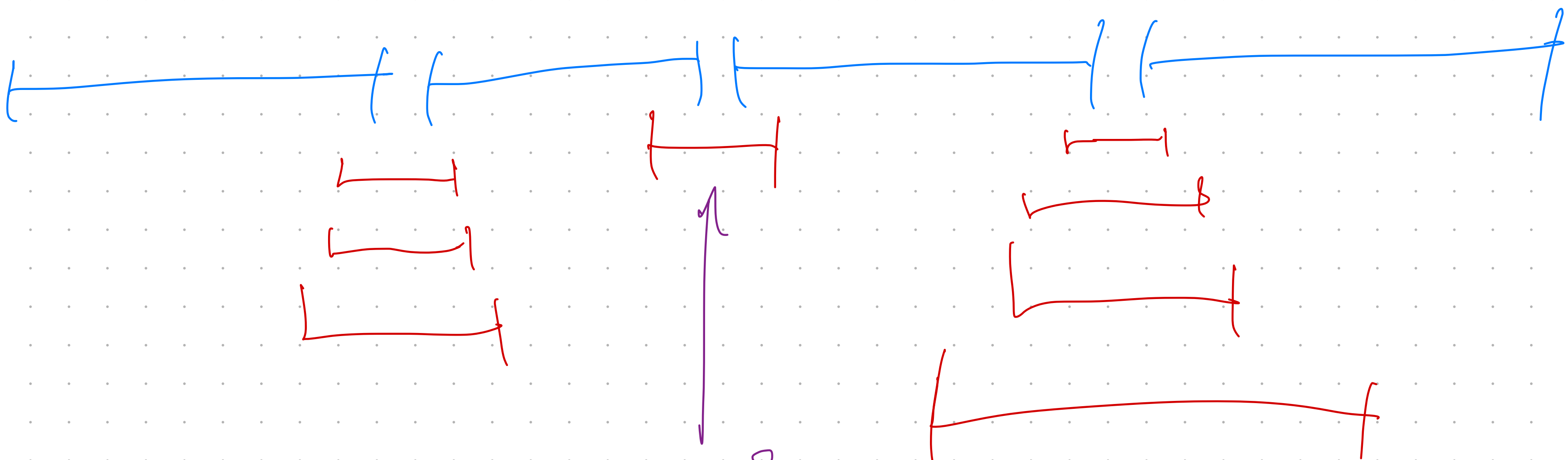


1. Earliest Start time

2. Shortest Interval



3. # of conflicts



ONLY 2
conflicts

Earliest Finish Time

1. Sort jobs by finish time
2. Schedule = { }

Iterate through jobs in sorted order $j = 1 \dots n$

- if job j does not conflict w/ Schedule
  - $\hookrightarrow$ Schedule $\leftarrow$ Schedule $\cup \{j\}$

Return Schedule.

Earliest Finish Time

1. Sort jobs by finish time
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   Iterate through jobs in sorted order $j = 1 \ldots n$
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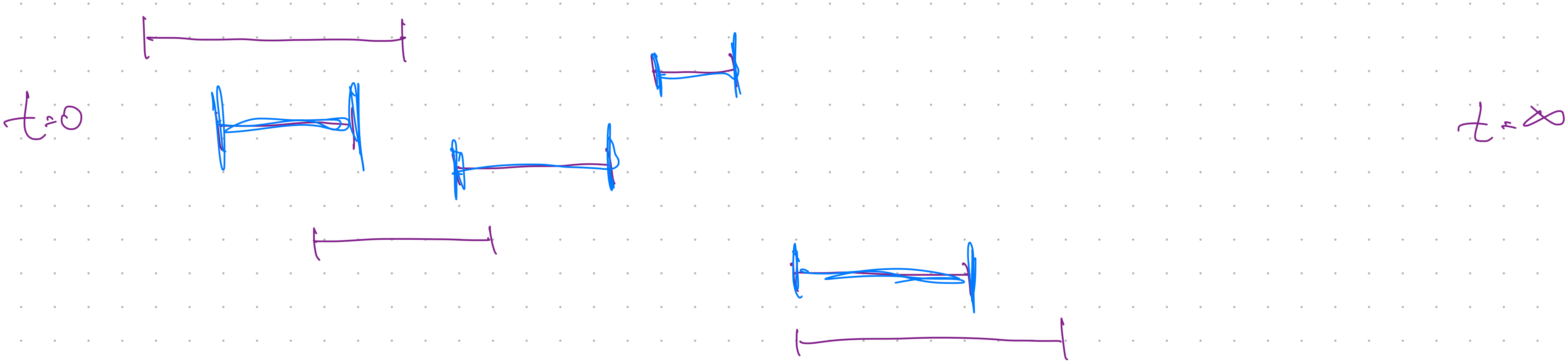

Earliest Finish Time

1. Sort jobs by finish time
2. Schedule = $\{\}$

   Iterate through jobs in sorted order $j = 1 \dots n$
   - if job j does not conflict w/ Schedule
     ↳ Schedule $\leftarrow$ Schedule $\cup \{j\}$

   Return Schedule.

Claim. EFT can be implemented with RT $O(n \log n)$.

Theorem. EFT returns a maximum cardinality set of non-conflicting jobs.

Non-conflicting ⟶ By design

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Maximum Cardinality ⟶ "Greedy stays ahead"

Greedy Stays ahead

- Imagine some optimal schedule O^*
- Compare output of EFT to O^*

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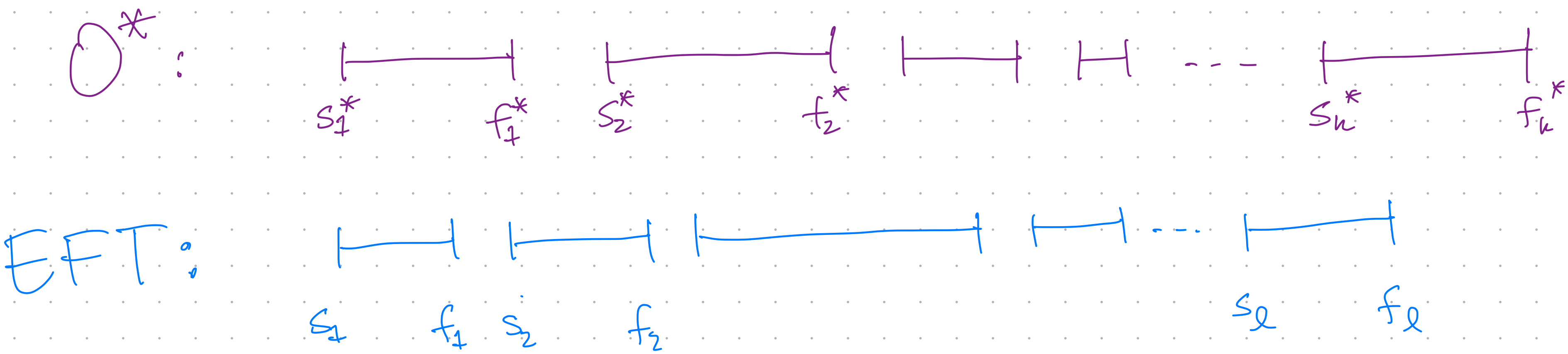
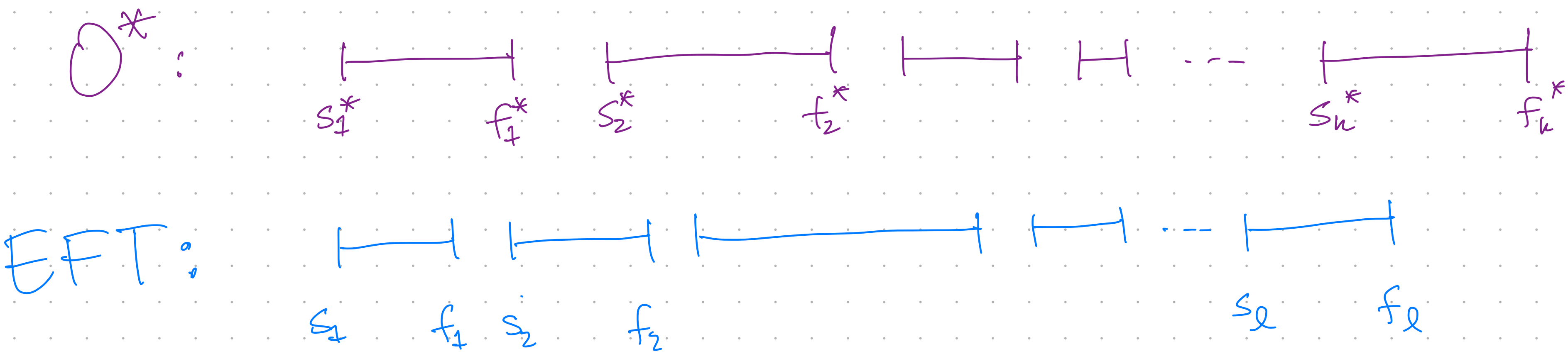


Greedy Stays ahead

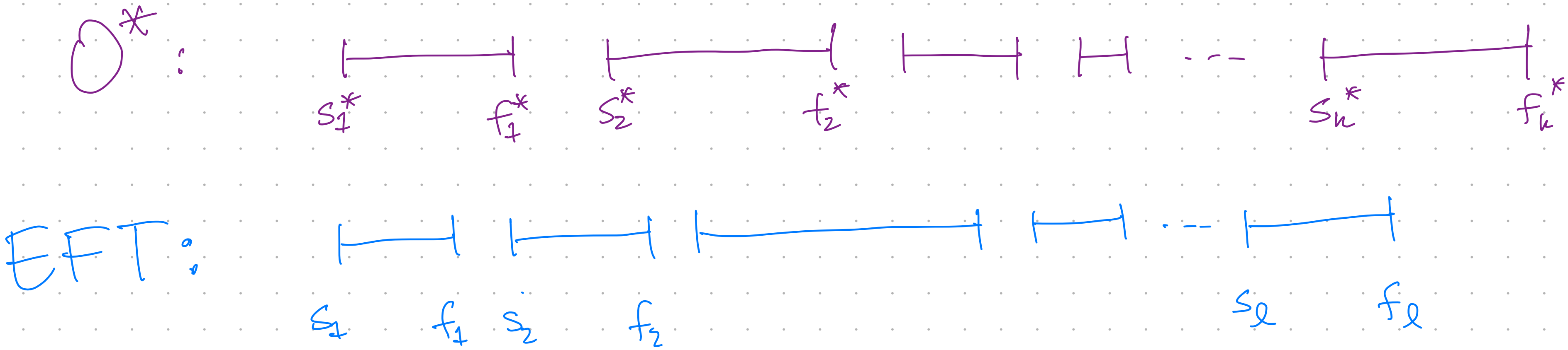
- Imagine *some* optimal schedule O^*
- Compare output of EFT to O^*



Convenient Notation. Assume jobs are sorted by *finishing* time.

$f_1^* < f_2^* < \dots < f_k^*$

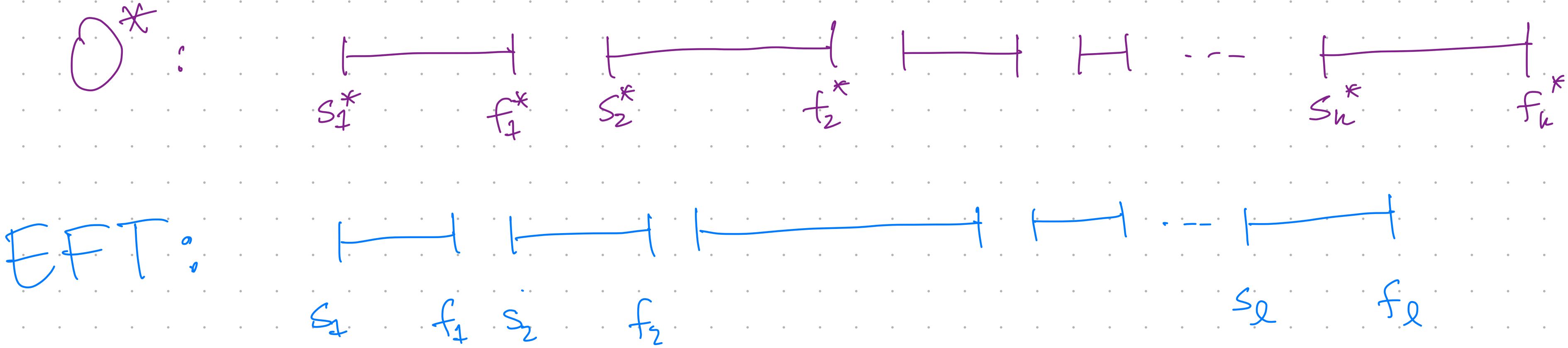
$f_1 < f_2 < \dots < f_\ell$



Argue that:

(a) if EFT "stays ahead" of O^*, then EFT is also optimal

(b) EFT "stays ahead"

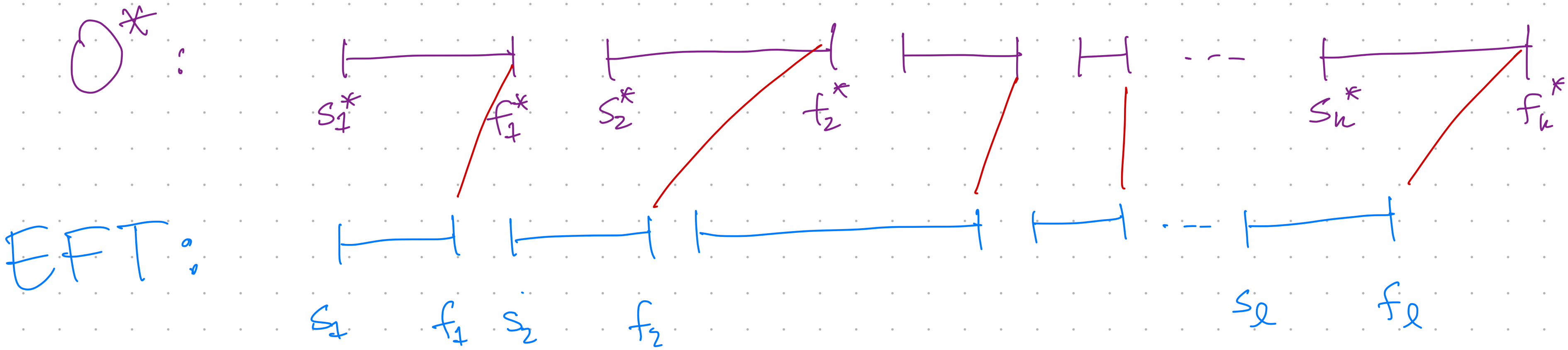


Argue that:

(a) if EFT "stays ahead" of O^*,
then EFT is also optimal

(b) EFT "stays ahead"

WARNING: Need to define "stays ahead" per problem



Claim. EFT "stays ahead" of any O^* in the finish time of the j^{th} job.

Optimal schedule $O^* = \langle [s_1^*, f_1^*], [s_2^*, f_2^*], \ldots, [s_\ell^*, f_\ell^*] \rangle$

EFT schedule $S = \langle [s_1, f_1], [s_2, f_2], \ldots, [s_k, f_k] \rangle$

<u>Greedy Stays Ahead Lemma.</u> (part (b))

Let S be the schedule returned by EFT.

For any optimal O^*,

$$\underbrace{\text{For all } j \in \{1, \ldots, |S|\} \quad f_j \leq f_j^*}_{\text{"EFT stays ahead of } O^*\text{"}}$$

Greedy Stays Ahead Lemma. (part (b))

Let S be the schedule returned by EFT.

For any optimal O^*,

$$\text{For all } j \in \{1, \ldots, |S|\} \quad f_j \leq f_j^*$$

Pf. By induction on intervals added by EFT.

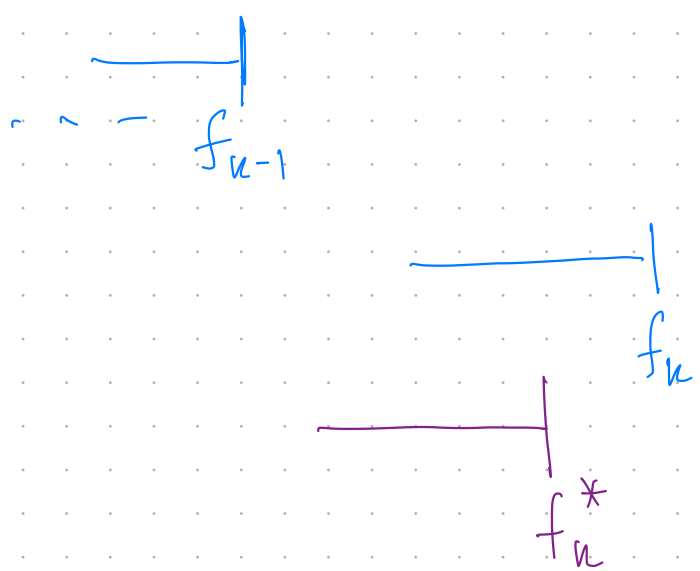
Greedy Stays Ahead Lemma. (part (b))

Let S be the schedule returned by EFT.
For any optimal O^*,

For all $j \in \{1, \dots, |S|\}$ $f_j \leq f_j^*$.

Pf. By induction on intervals added by EFT.

Induction step. (hypothesis holds for all $k' < k$)



why is this not possible?

Optimal schedule $O^* = \langle [s_1^*, f_1^*], [s_2^*, f_2^*], \ldots, [s_\ell^*, f_\ell^*] \rangle$

EFT schedule $S = \langle [s_1, f_1], [s_2, f_2], \ldots, [s_k, f_k] \rangle$

<u>Earliest Finish Time Lemma</u> [part (a)]

If for all $j \in \{1, \ldots, k\}$ $\quad f_j \leq f_j^*$

Then S is also <u>optimal</u>.

"If EFT stays ahead, then EFT is optimal"

Earliest Finish Time Lemma [part (a)]

If for all $j \in \{1, \ldots, k\}$ $\quad f_j \leq f_j^*$

Then S is also <u>optimal</u>.

<u>Pf</u>: By contradiction.

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If for all $j \in \{1, \ldots, k\}$ $f_j \leq f_j^*$

Then S is also optimal.

Pf: By contradiction.