26 Jan

Analyzing Gale-Shapley

The Proposal Algorithm

Initialize $M = \emptyset$

while $\exists$ a free firm f that hasn't
yet proposed to every applicant:
f finds its most preferred
applicant that it hasn't
yet proposed to, a.
if a is free:
insert (a, f) into M
if a is matched to some $f' \neq f$:
if a prefers f to f':
remove (a, f') from M
insert (a, f)
else:
do nothing

Output M

Obs 1. Firms make offers to applicants consecutively ordered in decreasing priority order on their preference list.

(If f ever makes an offer to a, f already offered to every applicant it prefers over a.)

Obs. 2 After an applicant receives their first offer, they remain matched for the remainder of the algorithm's execution. The sequence of firms they are matched with monotonically improves in their preference ordering.

Termination: $\leq n^2$ iterations because each offer involves a new pair.

Matching: Obvious, by the structure of the algorithm.

In more detail, induction on # of loop iterations.

INDUCTION HYPOTHESIS: At the start + end of each loop iteration, M is a matching.

BASE CASE: $M = \emptyset$ is a matching.

INDUCTION STEP: If we start an iteration with M being a matching, we insert at most one pair, (a, f). Furthermore if we do so, f was free, and either a was free or it belonged to a pair (a, f') that was deleted from M in the same iteration.

Perfect Matching: At termination, either every firm is matched, or some firm is free but has proposed to all applicants. Then by Obs. 2, every applicant is matched. This contradicts the assumption that a firm is free.

Stable. Consider (f, a) that are not in M at termination time. We must show they are not a blocking pair.

We must show either:

- f prefers its partner in M over a,
- a " " " " " over f.

If f proposed to a:

a rejected f's offer or accepted f but then accepted a better offer.
Either way, a prefers its partner over f.

Else:

f never reached a on its pref list
$\Rightarrow$ f prefers its partner to a.