Here are proofs of (9.14)-(9.18), p. 50. We'll use the notation of Supplementary Lecture A and refer to the axioms (A.1)-(A.13) instead of (9.1)-(9.13). First let's establish some useful lemmas.

Lemma 1 The operations $+$, $\cdot$, and * are *monotone* with respect to $\leq$; that is, for all a, b, c, if $a \leq b$, then

(i) $a + c \leq b + c$,
(ii) $c + a \leq c + b$,
(iii) $ac \leq bc$,
(iv) $ca \leq cb$,
(v) $a^* \leq b^*$.

To show (i),

$$\begin{aligned}
a \leq b &\Rightarrow a + b = b && \text{definition of } \leq \\
&\Rightarrow a + b + c = b + c \\
&\Rightarrow a + c + b + c = b + c && \text{idempotence, commutativity of +} \\
&\Rightarrow a + c \leq b + c && \text{definition of } \leq.
\end{aligned}$$

Property (ii) follows from (i) and commutativity of +. To show (iii),

$$\begin{aligned}
a \leq b &\Rightarrow a + b = b && \text{definition of } \leq \\
&\Rightarrow (a + b)c = bc \\
&\Rightarrow ac + bc = bc && \text{distributivity} \\
&\Rightarrow ac \leq bc && \text{definition of } \leq.
\end{aligned}$$

The argument for (iv) is symmetric. For (v), we have $1 + ab^* \leq 1 + bb^* = b^*$ by monotonicity of multiplication and (A.10). Then $a^* \leq b^*$ follows from (A.12).

Lemma 2 $a^*a^* = a^*$.

To show the inequality $\leq$, by (A.12) it suffices to show $a^* + aa^* \leq a^*$.

$$\begin{aligned}
a^* + aa^* &= 1 + aa^* + aa^* && \text{by (A.10)} \\
&= 1 + aa^* && \text{by idempotence} \\
&= a^* && \text{by (A.10).}
\end{aligned}$$

To show the reverse inequality, by (A.12) it suffices to show $1 + aa^*a^* \leq a^*a^*$.

$$\begin{aligned}
1 + aa^*a^* &\leq 1 + aa^* + aa^*a^* \\
&= a^* + aa^*a^* && \text{(A.10)} \\
&= (1 + aa^*)a^* && \text{distributivity} \\
&= a^*a^* && \text{(A.10).}
\end{aligned}$$

Lemma 3 $a^{**} = a^*$.

To show $\leq$, by (A.12) it suffices to show $1 + a^*a^* \leq a^*$. We have $1 \leq a^*$ by (A.10) and $a^*a^* \leq a^*$ by Lemma 2, thus $a^{**} \leq a^*$ since $+$ gives the least upper bound with respect to $\leq$.

For the reverse inequality, we know $a \leq a^*$, since by (A.10) and distributivity, $a^* = 1 + aa^* = 1 + a(1 + aa^*) = 1 + a + aaa^*$. Then $a^* \leq a^{**}$ follows from monotonicity of *.

(9.14) To prove $(ab)^*a = a(ba)^*$, it suffices to prove the two inequalities $(ab)^*a \le a(ba)^*$ and $(ab)^*a \ge a(ba)^*$. To show $(ab)^*a \le a(ba)^*$, by (A.12) it suffices to show $a + aba(ba)^* \le a(ba)^*$. But

$$\begin{aligned} a + aba(ba)^* &= a(1 + ba(ba)^*) && \text{distributivity (A.8)} \\ &= a(ba)^* && \text{(A.10).} \end{aligned}$$

The argument for the reverse inequality is symmetric, using (A.13), (A.9), and (A.11).

(9.15) To prove $(a^*b)^*a^* = (a+b)^*$, again it suffices to prove inequalities in both directions.

To show $(a^*b)^*a^* \le (a+b)^*$,

$$\begin{aligned} (a^*b)^*a^* &\le ((a+b)^*(a+b))^*(a+b)^* && \text{monotonicity} \\ &\le (a+b)^{**}(a+b)^* && \text{(A.11) and monotonicity} \\ &= (a+b)^*(a+b)^* && \text{Lemma 3} \\ &= (a+b)^* && \text{Lemma 2.} \end{aligned}$$

For the reverse inequality, by (A.12) it suffices to show $1 + (a+b)(a^*b)^*a^* \le (a^*b)^*a^*$. By distributivity and the fact that $+$ gives the least upper bound with respect to $\le$, it suffices to show

(a) $1 \le (a^*b)^*a^*$

(b) $a(a^*b)^*a^* \le (a^*b)^*a^*$

(c) $b(a^*b)^*a^* \le (a^*b)^*a^*$.

The inequality (a) follows from two applications of (A.10) and monotonicity. For (b),

$$\begin{aligned} a(a^*b)^*a^* &= a(1 + a^*b(a^*b)^*)a^* && \text{(A.10)} \\ &= aa^* + aa^*b(a^*b)^*a^* && \text{distributivity (A.8) and (A.9)} \\ &\le a^* + a^*b(a^*b)^*a^* && \text{(A.10) and monotonicity} \\ &= (1 + a^*b(a^*b)^*)a^* && \text{distributivity (A.9)} \\ &= (a^*b)^*a^* && \text{(A.10).} \end{aligned}$$

Finally, for (c), by (A.10) and monotonicity we have

$$b(a^*b)^*a^* \le a^*b(a^*b)^*a^* \le (a^*b)^*a^*.$$

(9.16) This follows immediately from (9.14) and (9.15).

(9.17) The inequality $\ge$ is immediate from monotonicity. For $\le$, using distributivity, commutativity, (A.10), and idempotence, we have

$$\begin{aligned} 1 + (1+a)a^* &= 1 + a^* + aa^* \\ &= 1 + aa^* + a^* \\ &= a^* + a^* \\ &= a^*, \end{aligned}$$

therefore $(1+a)^* \le a^*$ by (A.12).

(9.18) For $\le$, by distributivity and (A.11) we have $a + a^*aa = (1 + a^*a)a = a^*a$, therefore $aa^* \le a^*a$ by (A.13). The argument for the reverse inequality is symmetric.