The Perceptron

Cornell CS 4/5780

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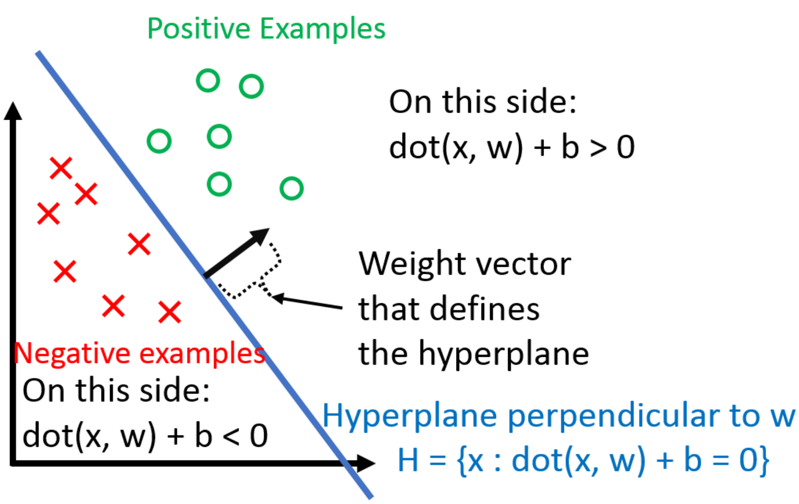
Assumptions

1. Binary classification (i.e. $y_i \in \{-1, +1\}$)
2. Data is linearly separable

Classifier

$$h(\mathbf{x}_i) = \textrm{sign}(\mathbf{w}^\top \mathbf{x}_i + b)$$



b is the bias term (without the bias term, the hyperplane that $\mathbf{w}$ defines would always have to go through the origin). Dealing with b can be a pain, so we 'absorb' it into the feature vector $\mathbf{w}$ by adding one additional *constant* dimension. Under this convention,

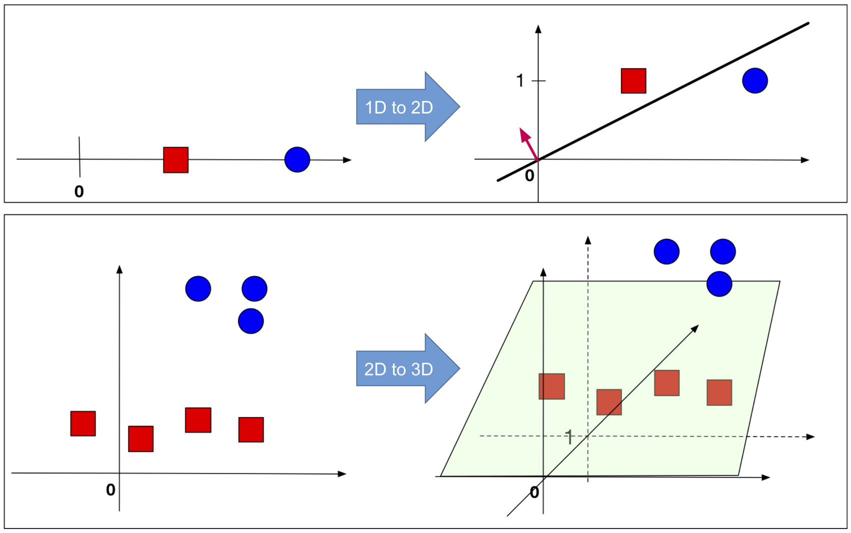
$$\mathbf{x}_i \textrm{ becomes } \begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix} \mathbf{w} \textrm{ becomes } \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix}$$

We can verify that

$$\begin{bmatrix} \mathbf{x}_i \\ 1 \end{bmatrix}^\top \begin{bmatrix} \mathbf{w} \\ b \end{bmatrix} = \mathbf{w}^\top \mathbf{x}_i + b$$

Using this, we can simplify the above formulation of $h(\mathbf{x}_i)$ to

$$h(\mathbf{x}_i) = \textrm{sign}(\mathbf{w}^\top \mathbf{x})$$



(Left:) The original data is 1-dimensional (top row) or 2-dimensional (bottom row). There is no hyper-plane that passes through the origin and separates the red and blue points. (Right:) After a constant dimension was added to all data points such a hyperplane exists.

<u>Observation:</u> Note that

$$y_i(\mathbf{w}^\top \mathbf{x}_i) > 0 \Longleftrightarrow \mathbf{x}_i \text{ is classified correctly}$$

where 'classified correctly' means that x_i is on the correct side of the hyperplane defined by $\mathbf{w}$. Also, note that the left side depends on $y_i \in \{-1, +1\}$ (it wouldn't work if, for example $y_i \in \{0, +1\}$).

Perceptron Algorithm

Now that we know what the $\mathbf{w}$ is supposed to do (defining a hyperplane the separates the data), let's look at how we can get such $\mathbf{w}$. **Perceptron Algorithm**

Initialize $\vec{w} = \vec{0}$	// Initialize $\vec{w}$. $\vec{w} = \vec{0}$ misclassifies everything.
while TRUE **do**	// Keep looping
$m = 0$	// Count the number of misclassifications, m
for $(x_i, y_i) \in D$ **do**	// Loop over each (data, label) pair in the dataset, D
if $y_i(\vec{w}^T \cdot \vec{x_i}) \leq 0$ **then**	// If the pair $(\vec{x_i}, y_i)$ is misclassified
$\vec{w} \leftarrow \vec{w} + y\vec{x}$	// Update the weight vector $\vec{w}$
$m \leftarrow m + 1$	// Counter the number of misclassification
end if	
end for	
if $m = 0$ **then**	// If the most recent $\vec{w}$ gave 0 misclassifications
break	// Break out of the while-loop
end if	
end while	// Otherwise, keep looping!

Geometric Intuition

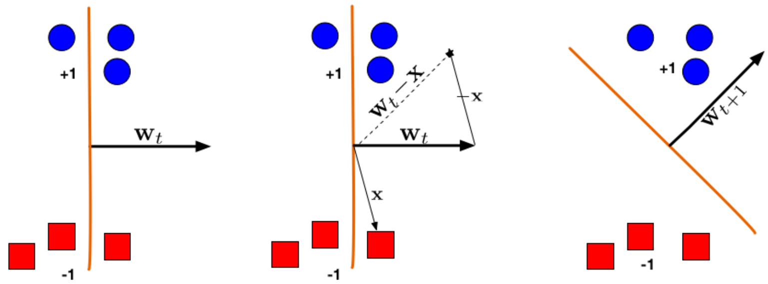


Illustration of a Perceptron update. (Left:) The hyperplane defined by $\mathbf{w}_t$ misclassifies one red (-1) and one blue (+1) point. (Middle:) The red point $\mathbf{x}$ is chosen and used for an update. Because its label is -1 we need to ***subtract*** $\mathbf{x}$ *from $\mathbf{w}_t$. (Right:) The udpated hyperplane $\mathbf{w}_{t+1} = \mathbf{w}_t - \mathbf{x}$ separates the two classes and the Perceptron algorithm has converged.*

Quiz: Assume a data set consists only of a single data point $\{(\mathbf{x}, +1)\}$. How often can a Perceptron misclassify this point $\mathbf{x}$ repeatedly? What if the initial weight vector $\mathbf{w}$ was initialized randomly and not as the all-zero vector?

Perceptron Convergence

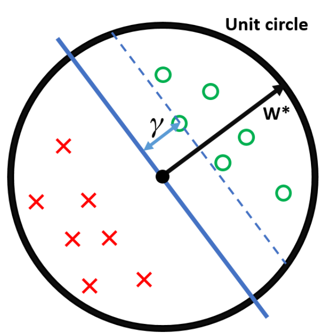
The Perceptron was arguably the first algorithm with a strong formal guarantee. If a data set is linearly separable, the Perceptron will find a separating hyperplane in a finite number of updates. (If the data is not linearly separable, it will loop forever.)

The argument goes as follows: Suppose $\exists \mathbf{w}^*$ such that $y_i(\mathbf{x}^\top \mathbf{w}^*) > 0$ $\forall (\mathbf{x}_i, y_i) \in D$. Now, suppose that we rescale each data point and the $\mathbf{w}^*$ such that

$$||\mathbf{w}^*|| = 1 \quad \text{and} \quad ||\mathbf{x}_i|| \leq 1 \ \forall \mathbf{x}_i \in D$$

Let us define the Margin γ of the hyperplane $\mathbf{w}^*$ as $\gamma = \min_{(\mathbf{x}_i, y_i) \in D} |\mathbf{x}_i^\top \mathbf{w}^*|$.

A little observation (which will come in very handy): For all $\mathbf{x}$ we must have $y(\mathbf{x}^\top \mathbf{w}^*) = |\mathbf{x}^\top \mathbf{w}^*| \geq \gamma$. Why? Because $\mathbf{w}^*$ is a perfect classifier, so all training data points $(\mathbf{x}, y)$ lie on the "correct" side of the hyper-plane and therefore $y = sign(\mathbf{x}^\top \mathbf{w}^*)$. The second inequality follows directly from the definition of the margin γ.



To summarize our setup:

- All inputs $\mathbf{x}_i$ live within the unit sphere
- There exists a separating hyperplane defined by $\mathbf{w}^*$, with $\|\mathbf{w}\|^* = 1$ (i.e. $\mathbf{w}^*$ lies exactly on the unit sphere).
- γ is the distance from this hyperplane (blue) to the closest data point.

Theorem: If all of the above holds, then the Perceptron algorithm makes at most $1/\gamma^2$ mistakes. **Proof:**
Keeping what we defined above, consider the effect of an update ($\mathbf{w}$ becomes $\mathbf{w} + y\mathbf{x}$) on the two terms $\mathbf{w}^\top \mathbf{w}^*$ and $\mathbf{w}^\top \mathbf{w}$. We will use two facts:

- $y(\mathbf{x}^\top \mathbf{w}) \leq 0$: This holds because $\mathbf{x}$ is misclassified by $\mathbf{w}$ - otherwise we wouldn't make the update.
- $y(\mathbf{x}^\top \mathbf{w}^*) > 0$: This holds because $\mathbf{w}^*$ is a separating hyper-plane and classifies all points correctly.

1. Consider the effect of an update on $\mathbf{w}^\top \mathbf{w}^*$:

$$(\mathbf{w} + y\mathbf{x})^\top \mathbf{w}^* = \mathbf{w}^\top \mathbf{w}^* + y(\mathbf{x}^\top \mathbf{w}^*) \geq \mathbf{w}^\top \mathbf{w}^* + \gamma$$

   The inequality follows from the fact that, for $\mathbf{w}^*$, the distance from the hyperplane defined by $\mathbf{w}^*$ to $\mathbf{x}$ must be at least γ (i.e. $y(\mathbf{x}^\top \mathbf{w}^*) = |\mathbf{x}^\top \mathbf{w}^*| \geq \gamma$). <u>This means that for each update</u>, $\mathbf{w}^\top \mathbf{w}^*$ <u>grows by **at least** γ</u>.
2. Consider the effect of an update on $\mathbf{w}^\top \mathbf{w}$:

$$(\mathbf{w} + y\mathbf{x})^\top (\mathbf{w} + y\mathbf{x}) = \mathbf{w}^\top \mathbf{w} + \underbrace{2y(\mathbf{w}^\top \mathbf{x})}_{<0} + \underbrace{y^2(\mathbf{x}^\top \mathbf{x})}_{0\leq \ \leq 1} \leq \mathbf{w}^\top \mathbf{w} + 1$$

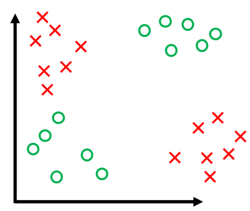
   The inequality follows from the fact that
   - $2y(\mathbf{w}^\top \mathbf{x}) < 0$ as we had to make an update, meaning $\mathbf{x}$ was misclassified
   - $0 \leq y^2(\mathbf{x}^\top \mathbf{x}) \leq 1$ as $y^2 = 1$ and all $\mathbf{x}^\top \mathbf{x} \leq 1$ (because $\|\mathbf{x}\| \leq 1$).

   <u>This means that for each update</u>, $\mathbf{w}^\top \mathbf{w}$ <u>grows by **at most** 1</u>.
3. Now remember from the Perceptron algorithm that we initialize $\mathbf{w} = \mathbf{0}$. Hence, initially $\mathbf{w}^\top \mathbf{w} = 0$ and $\mathbf{w}^\top \mathbf{w}^* = 0$ and after M updates the following two inequalities must hold:
   (1) $\mathbf{w}^\top \mathbf{w}^* \geq M\gamma$
   (2) $\mathbf{w}^\top \mathbf{w} \leq M$.
   We can then complete the proof:

$$\begin{aligned}
M\gamma &\leq \mathbf{w}^\top \mathbf{w}^* && \text{By (1)} \\
&= \|\mathbf{w}\| \cos(\theta) && \text{by definition of inner-product, where } \theta \text{ is the angle between } \mathbf{w} \text{ and} \\
&\leq ||\mathbf{w}|| && \text{by definition of cos, we must have } \cos(\theta) \leq 1. \\
&= \sqrt{\mathbf{w}^\top \mathbf{w}} && \text{by definition of } \|\mathbf{w}\| \\
&\leq \sqrt{M} && \text{By (2)} \\
& \\
&\Rightarrow M\gamma \leq \sqrt{M} \\
&\Rightarrow M^2\gamma^2 \leq M \\
&\Rightarrow M \leq \frac{1}{\gamma^2} && \text{And hence, the number of updates } M \text{ is bounded from above by a}
\end{aligned}$$

Quiz: Given the theorem above, what can you say about the margin of a classifier (what is more desirable, a large margin or a small margin?) Can you characterize data sets for which the Perceptron algorithm will converge quickly? Draw an example.

History



- Initially, huge wave of excitement ("Digital brains") (See The New Yorker December 1958)
- Then, contributed to the A.I. Winter. Famous example of a simple non-linearly separable data set, the XOR problem (Minsky 1969):