Solving Markov Decision Processes

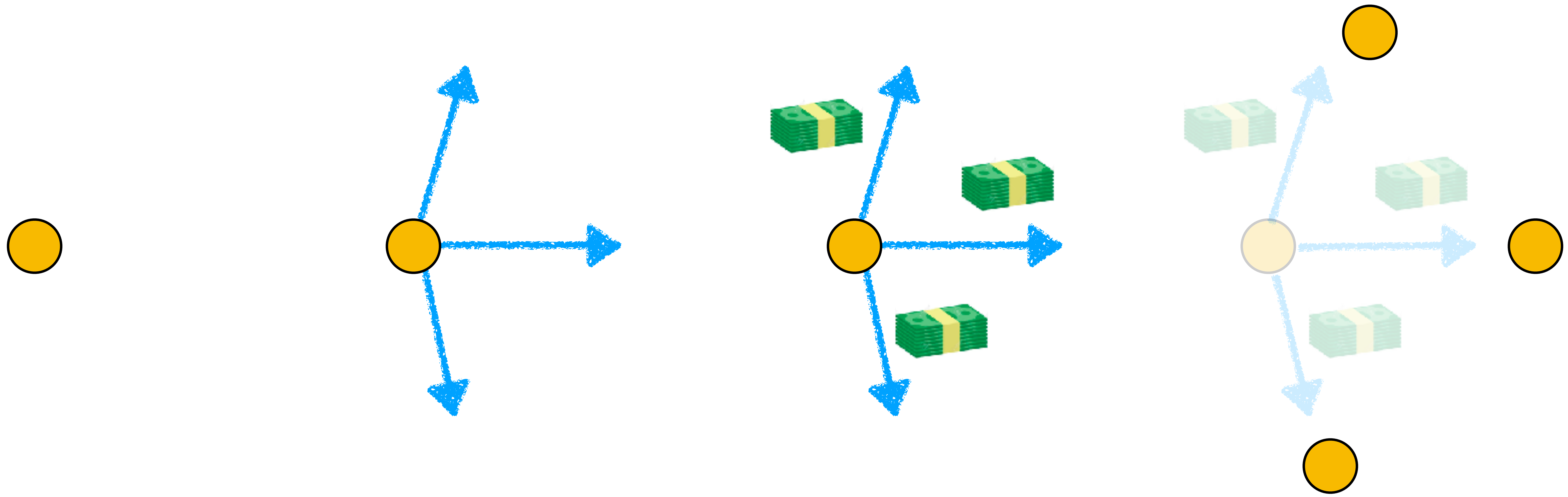
Sanjiban Choudhury

Cornell Bowers CIS Computer Science

Markov Decision Process

A mathematical framework for modeling sequential decision making

$$< S\ ,\ A\ ,\ C\ ,\ \mathcal{T} >$$

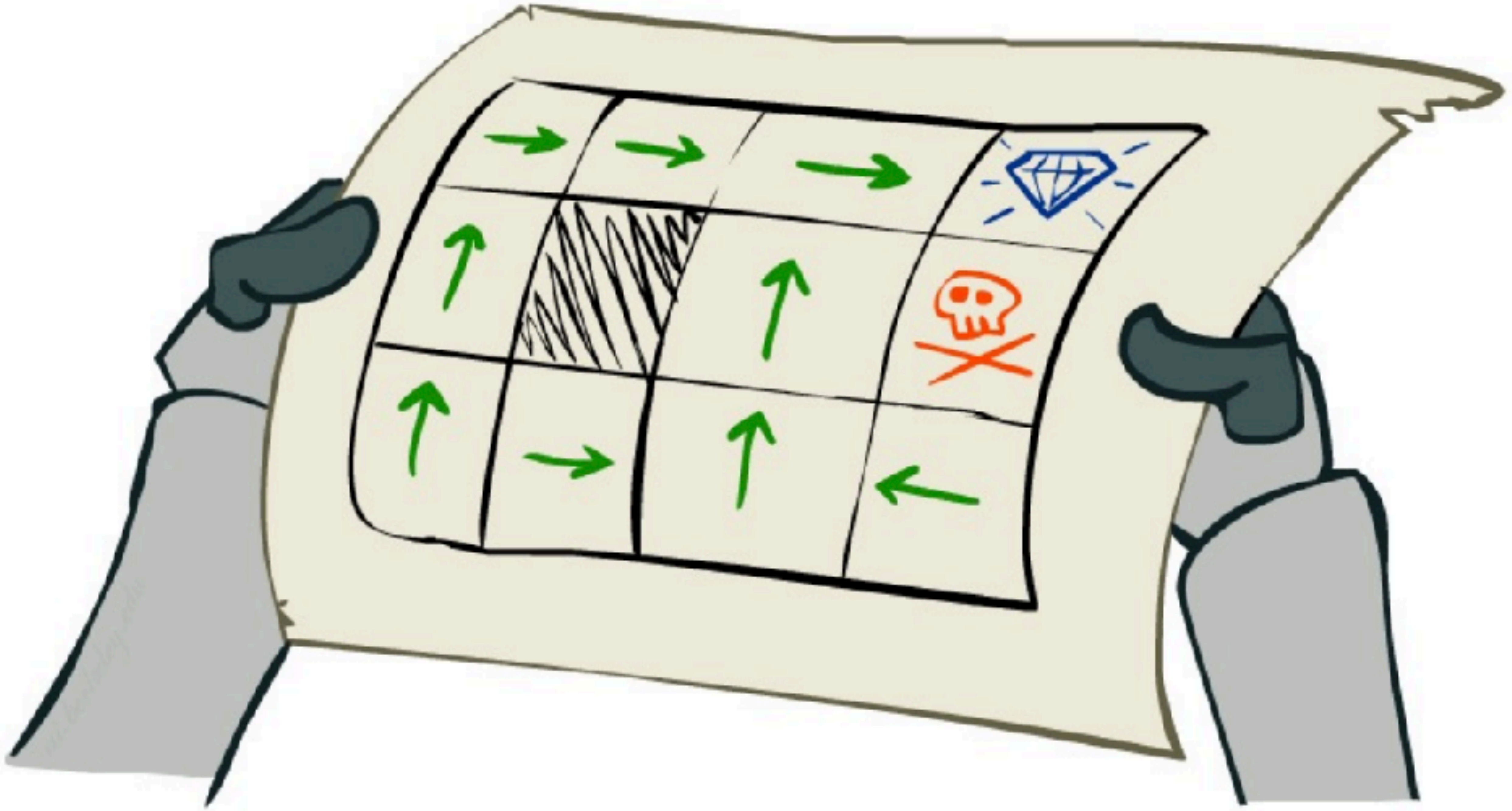


What does it mean to solve a MDP?

Solving an MDP means finding a Policy

$$\pi : s_t \rightarrow a_t$$

A function that maps state (and time) to action

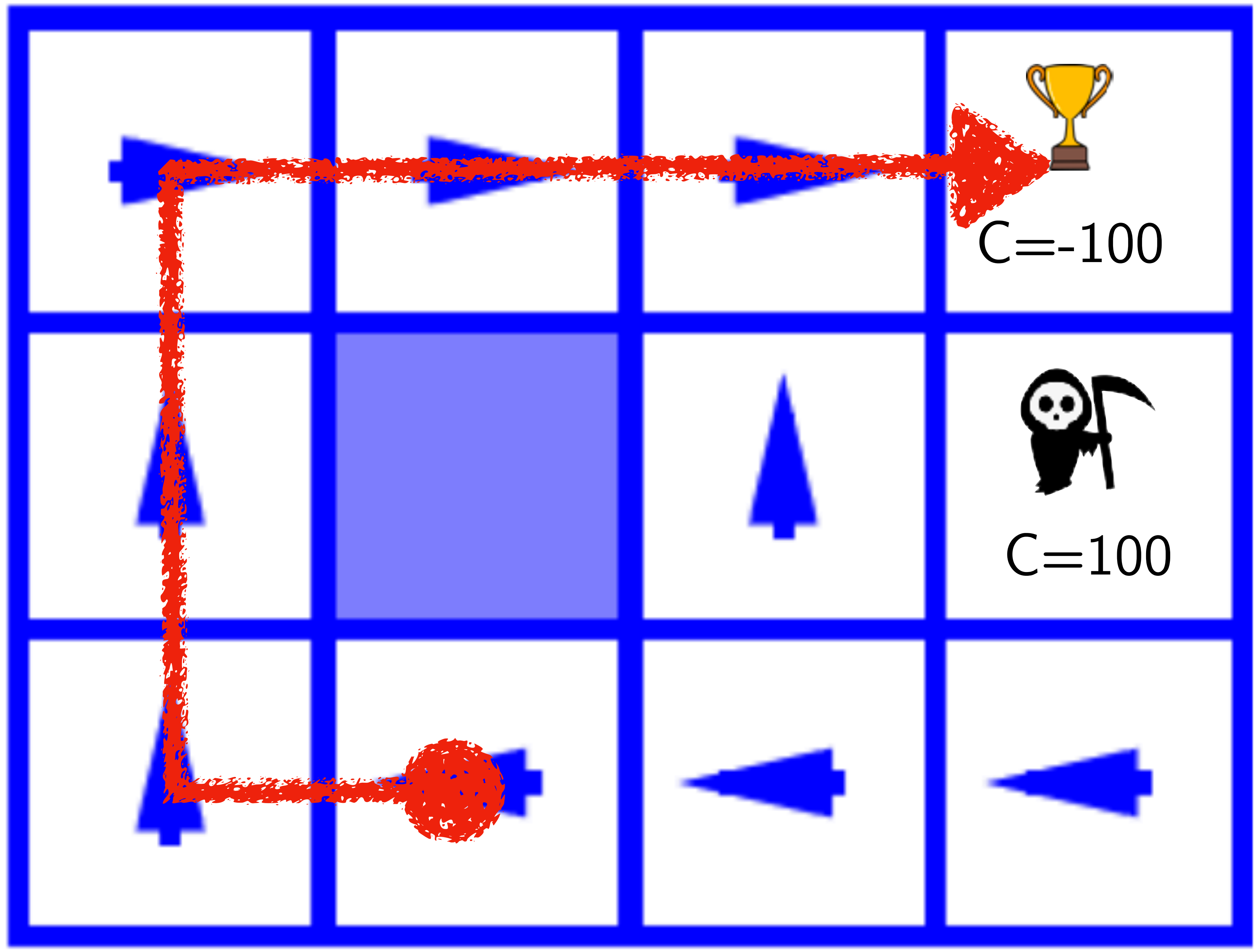


Policy: What action should I choose at any state?

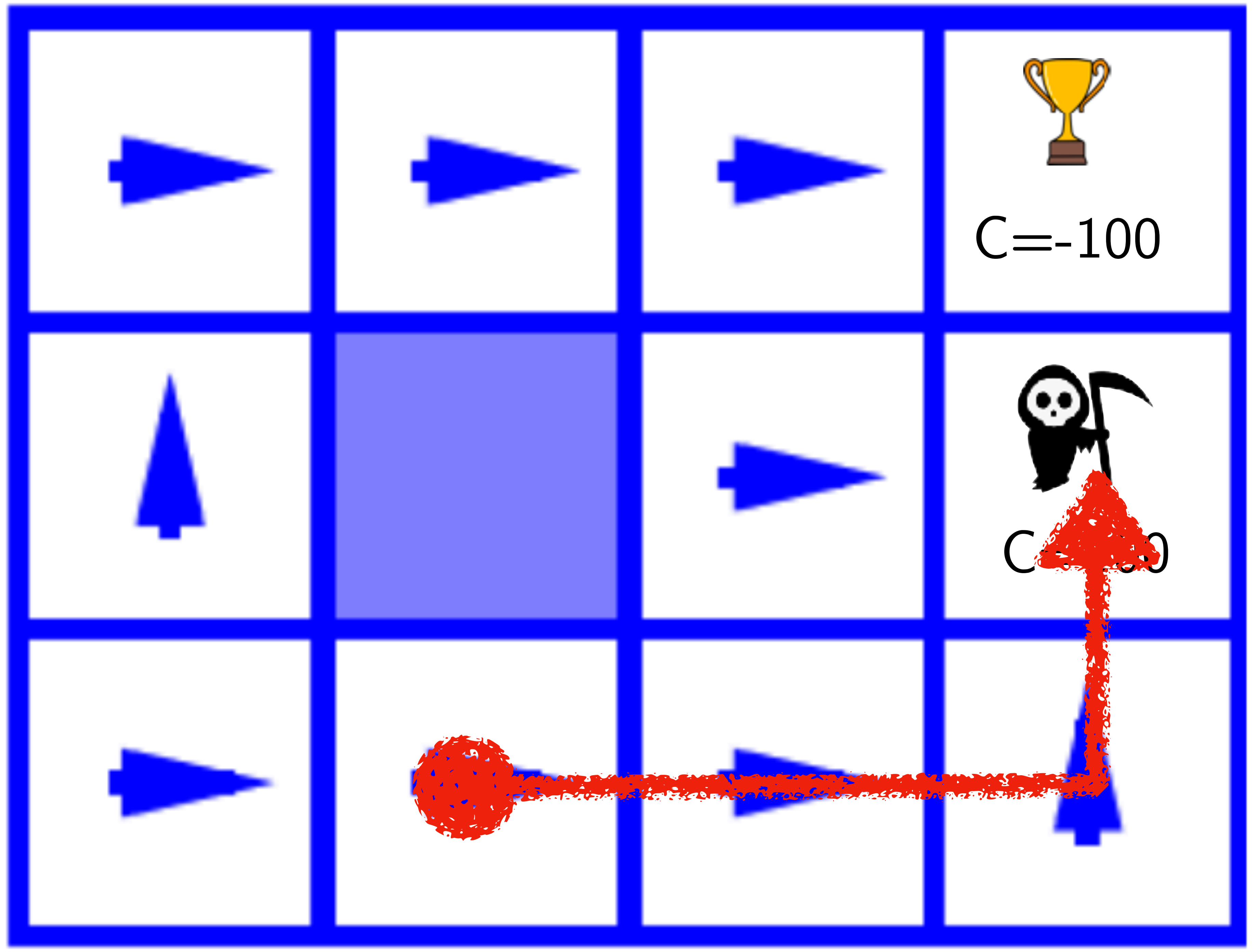
Image courtesy Dan Klein

What makes a policy *optimal?*

Which policy is better?



Policy π_1



Policy π_2

Courtesy Dan Klein

What makes a policy *optimal?*

$$\min_{\pi} \underset{\substack{a_t \sim \pi(s_t) \\ s_{t+1} \sim \mathcal{T}(s_t, a_t)}}{\mathbb{E}} \left[\sum_{t=0}^{T-1} c(s_t, a_t) \right]$$

(Search over Policies)

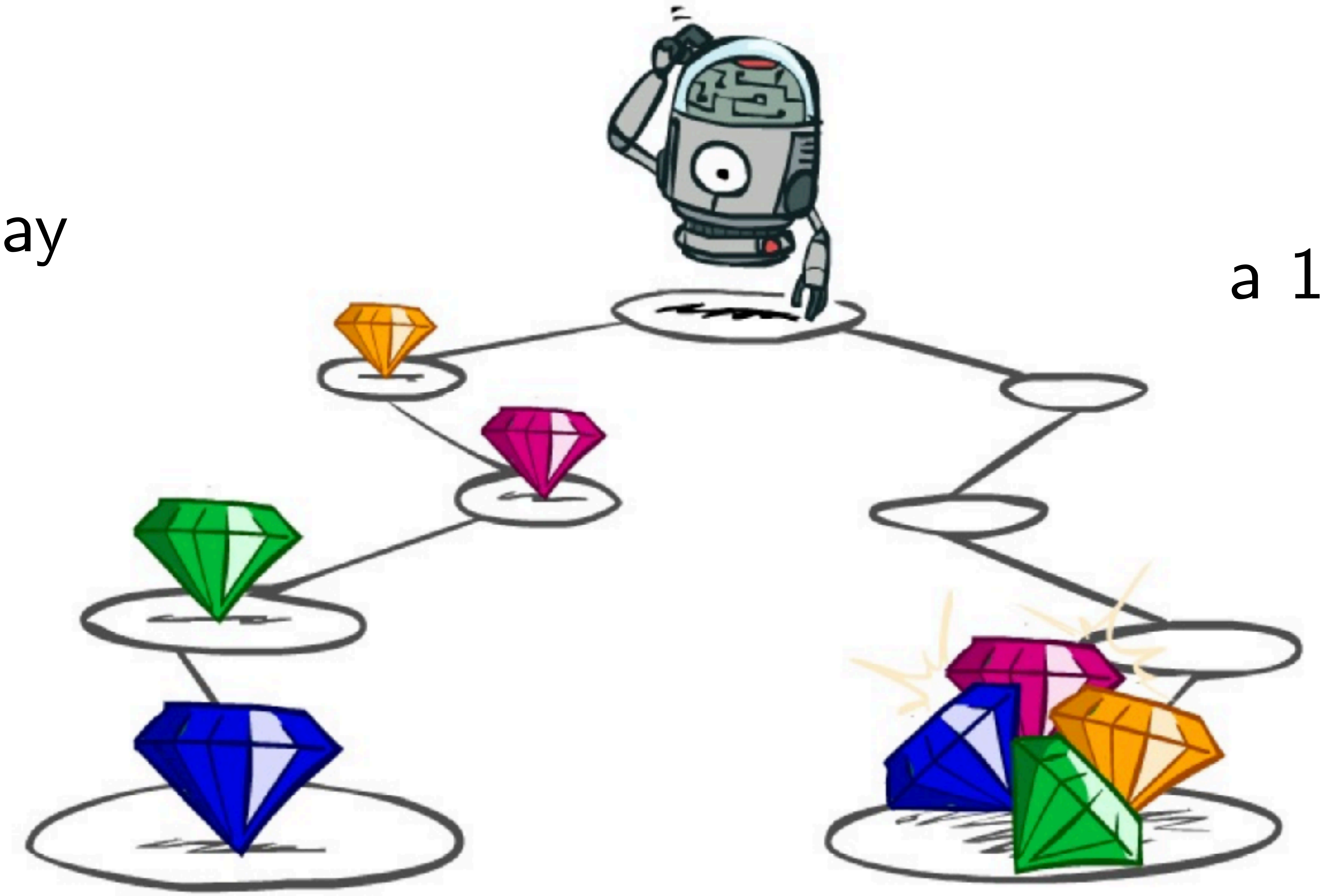
(Sample a start state, then follow π till end of episode)

(Sum over all costs)

One last piece ...

Which of the two outcomes do you prefer?

$50 today



$1 million
a 1000 days later

Image courtesy Dan Klein

Discount: Future rewards / costs matter less

At what discount value does it make sense to take $50 today than $1million in 1000 days?

Image courtesy Dan Klein

What makes a policy *optimal?*

$$\min_{\pi} \mathbb{E}_{\substack{a_t \sim \pi(s_t) \\ s_{t+1} \sim \mathcal{T}(s_t, a_t)}} \left[\sum_{t=0}^{T-1} \gamma^t c(s_t, a_t) \right]$$

(Search over Policies)

(Sample a start state, then follow π till end of episode)

(Discounted sum of costs)

How do we solve a MDP?

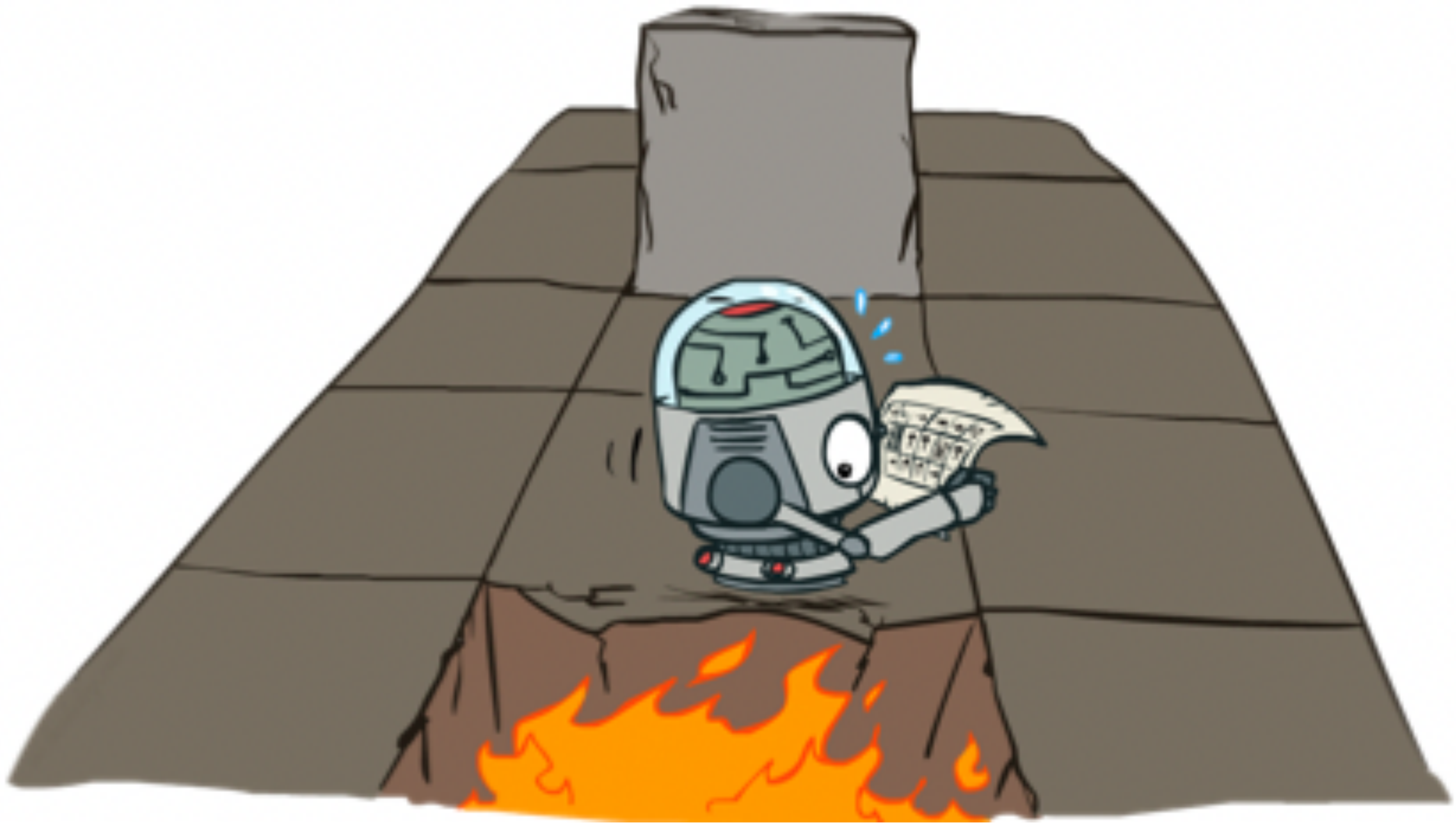


Image courtesy Dan Klein

Let's start with how NOT to solve MDPs

What would brute force do?

$$\min_{\pi} \mathop{\mathbb{E}}_{\substack{a_t \sim \pi(s_t) \\ s_{t+1} \sim \mathcal{T}(s_t, a_t)}} [\sum_{t=0}^{T-1} \gamma^t c(s_t, a_t)]$$

How much work would brute force have to do?

What would brute force do?

$$\min_{\pi} \mathbb{E}_{\substack{a_t \sim \pi(s_t) \\ s_{t+1} \sim \mathcal{T}(s_t, a_t)}} [\sum_{t=0}^{T-1} \gamma^t c(s_t, a_t)]$$

1. Iterate over all possible policies

2. For every policy, evaluate the cost

3. Pick the best one

There are $(A^S)^T$ Policies!!!!

MDPs have a very special structure

Introducing the "Value" Function

$$V^{\pi}(s_t)$$

Read this as: Value of a policy at a given state and time

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$$V^{\pi}(s_t)$$

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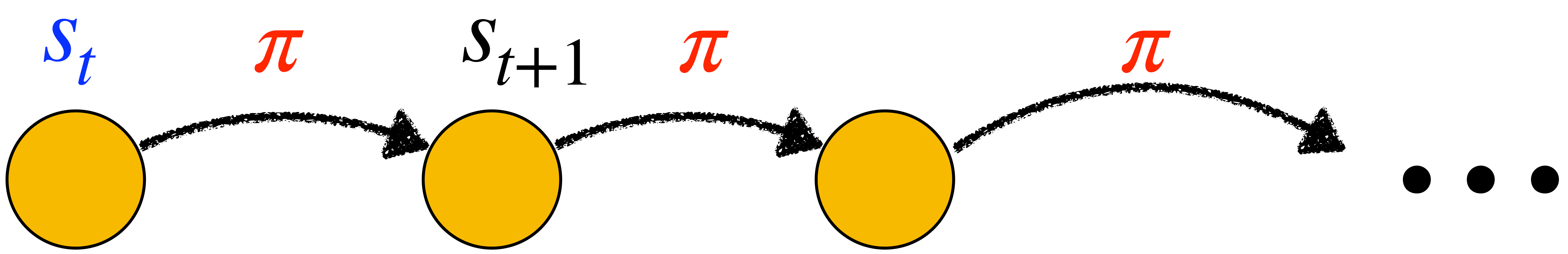


$$V^{\pi}(s_t) = c_t + \gamma c_{t+1} + \gamma^2 c_{t+2} + \dots$$

The Bellman Equation

$$V^{\pi}(s_t) = c(s_t, \pi(s_t)) + \gamma \mathbb{E}_{s_{t+1}} V^{\pi}(s_{t+1})$$

Value of current state

Cost

Value of future state

Why is this true?

Optimal policy

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{s_0} V^{\pi}(s_0)$$

Bellman Equation for the Optimal Policy

$$V^{\pi^*}(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} V^{\pi^*}(s_{t+1})) \right]$$

Optimal Value

Cost

Optimal Value of Next State

Why is this true?

We use V^* to denote optimal value

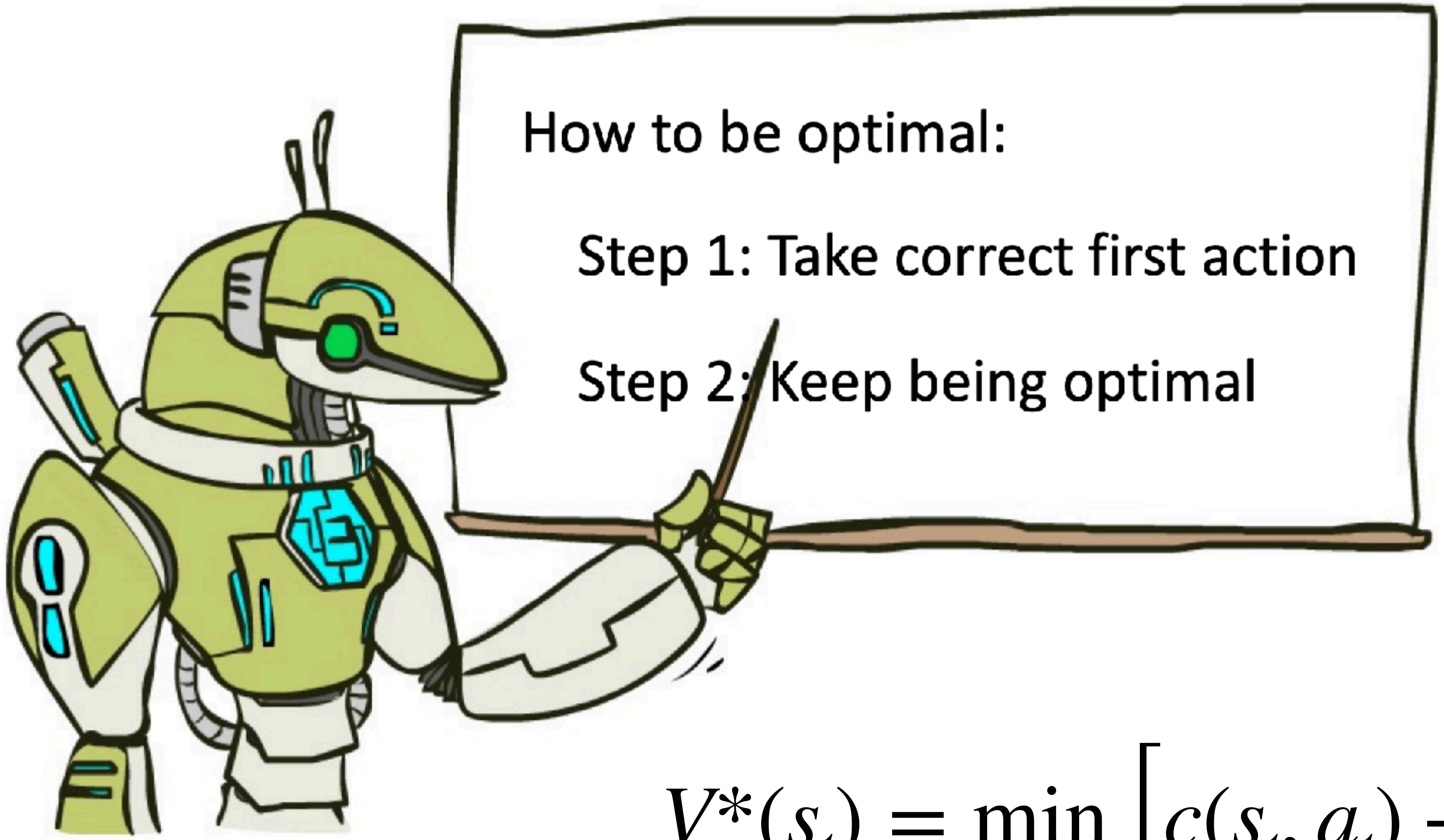
$$V^*(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} V^*(s_{t+1})) \right]$$

Optimal Value

Cost

Optimal Value of Next State

The Bellman Equation



$$V^*(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} V^*(s_{t+1})) \right]$$

Image courtesy Dan Klein

Activity!

Value Iteration

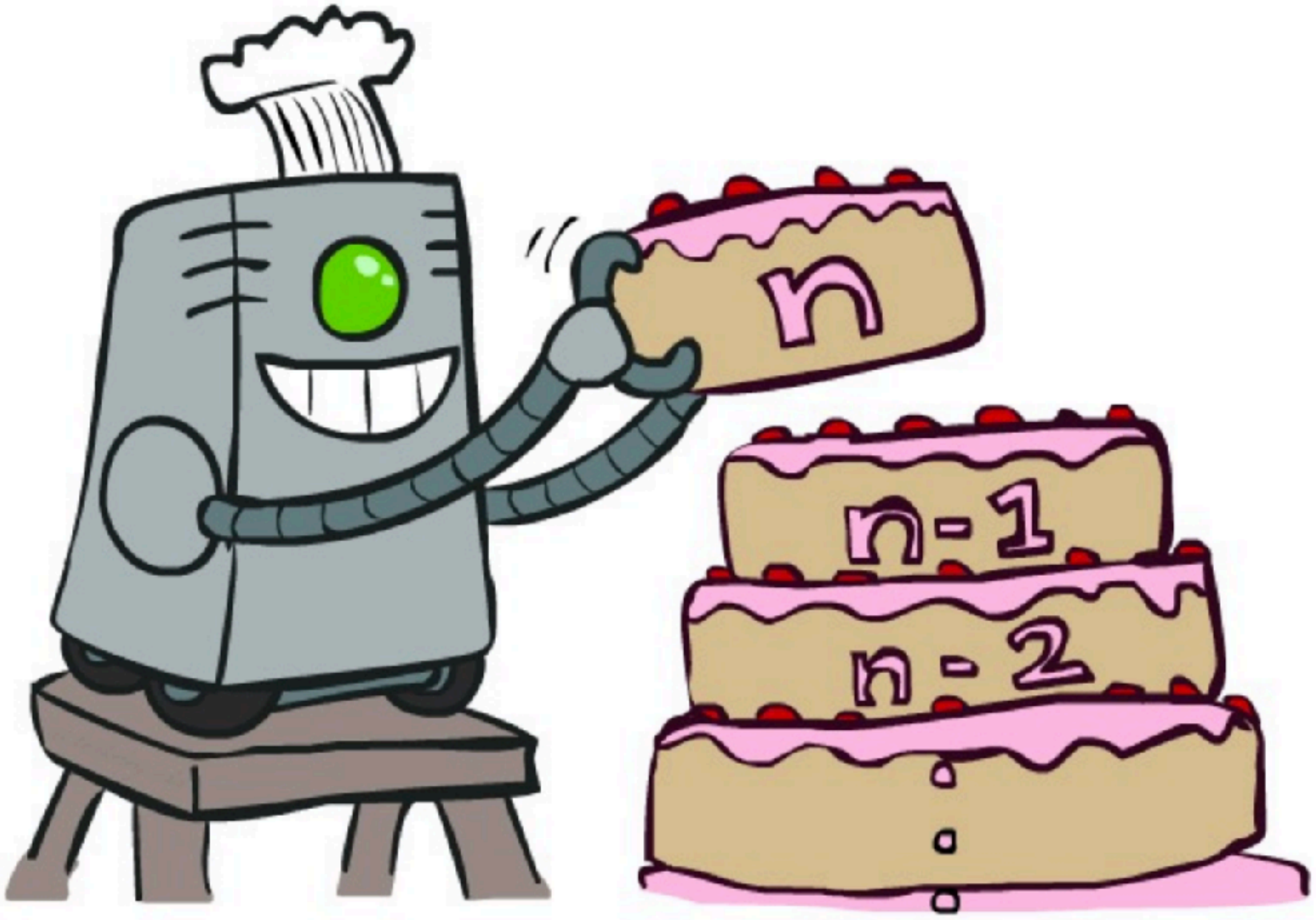
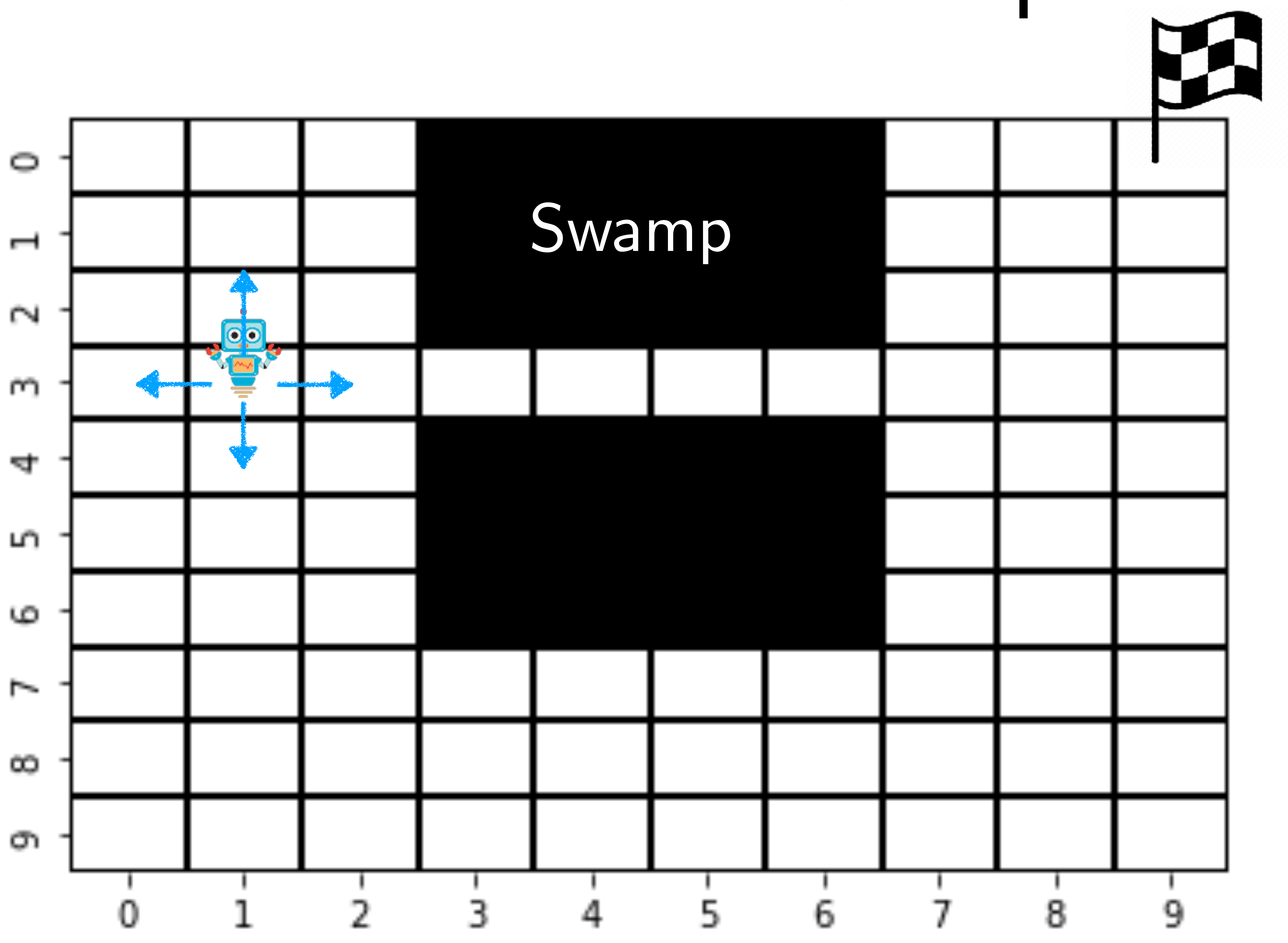


Image courtesy Dan Klein

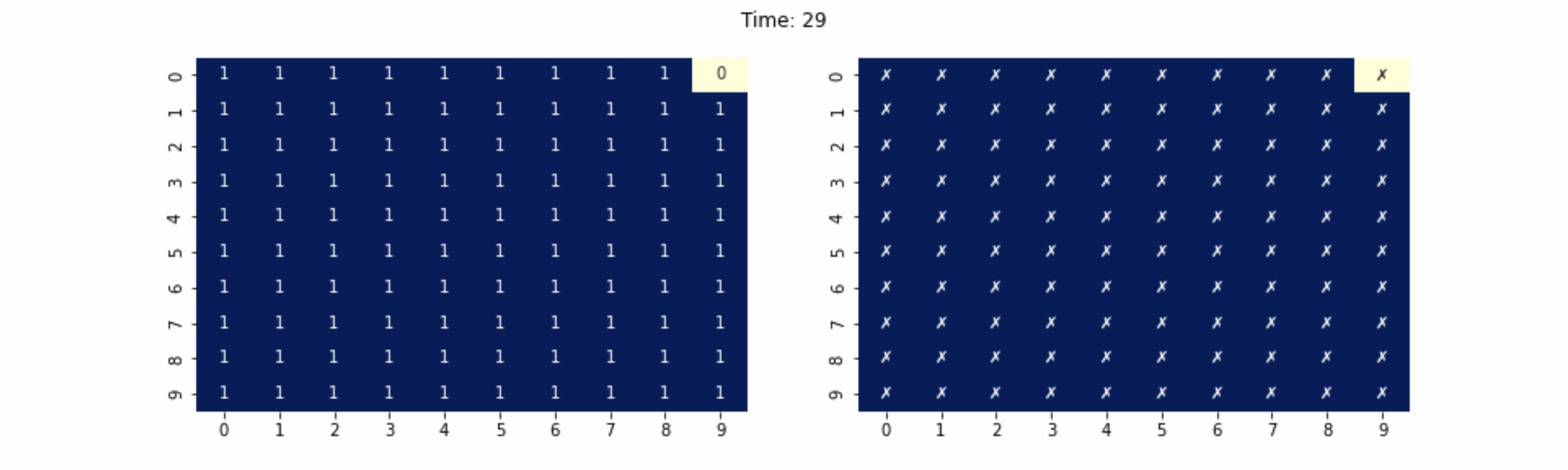
Setup



$< S, A, C, \mathcal{T} >$

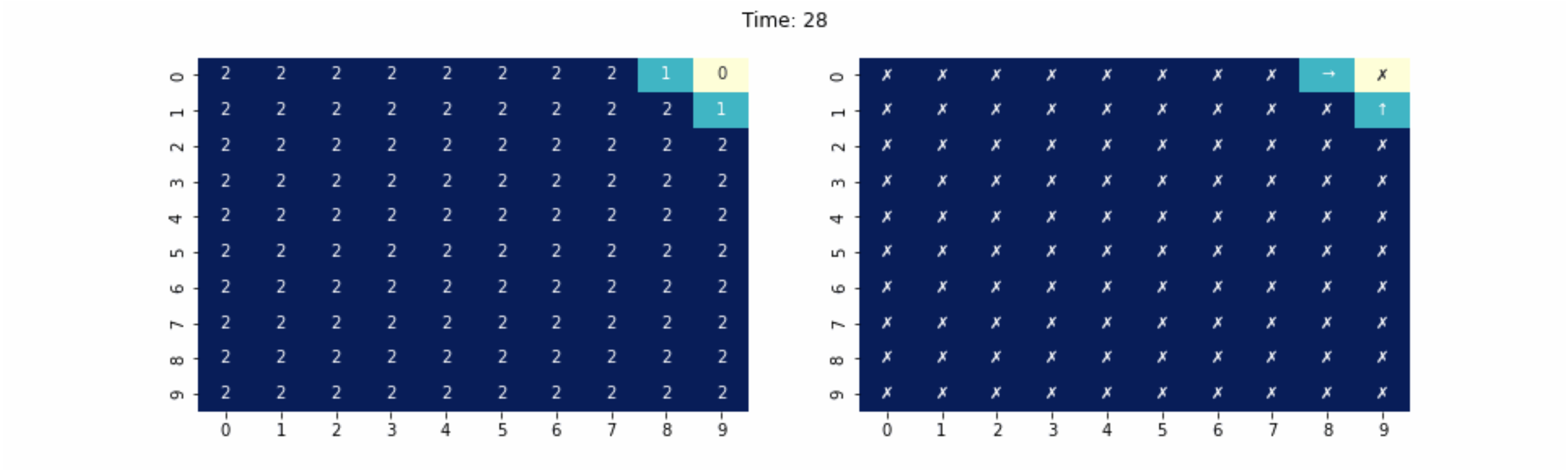
- Two absorbing states: Goal and Swamp (can never leave)
- c(s) = 0 at the goal, c(s) = 1 everywhere else
- Transitions deterministic
- Time horizon T = 30
- Discount $\gamma = 1$

What is the optimal value at T-1?



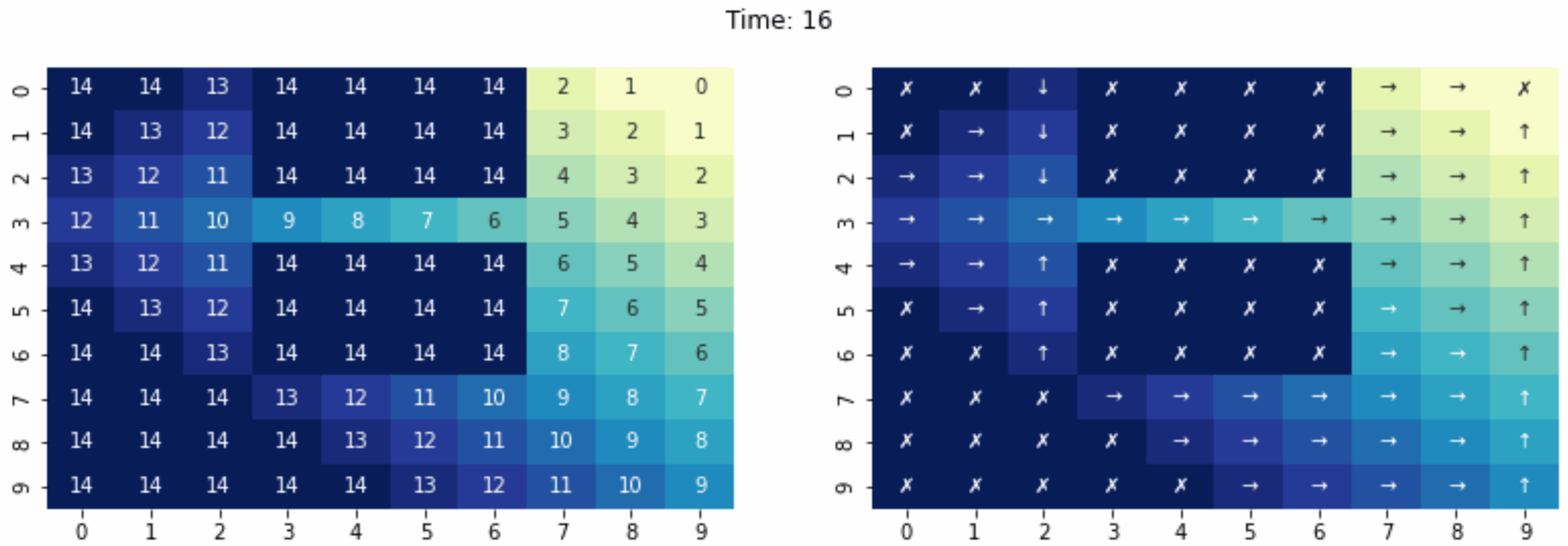
$$V^*(s_{T-1}) = \min_a c(s_{T-1}, a) \qquad \pi^*(s_{T-1}) = \arg\min_a c(s_{T-1}, a)$$

What is the optimal value at T-2?



$$V^*(s_{T-2}) = \min_a [c(s_{T-2}, a) + V^*(s_{T-1})] \qquad \pi^*(s_{T-2}) = \arg\min_a [c(s_{T-2}, a) + V^*(s_{T-1})]$$

Dynamic Programming all the way!



$$V^*(s_t) = \min_a [c(s_t, a) + V^*(s_{t+1})]$$

$$\pi^*(s_t) = \arg\min_a [c(s_t), a) + V^*(s_{t+1})]$$

Value Iteration

Initialize value function at last time-step

$$V^*(s, T-1) = \min_a c(s, a)$$

for $t = T-2, \ldots, 0$

Compute value function at time-step t

$$V^*(s, t) = \min_a \left[c(s, a) + \gamma \sum_{s'} \mathcal{T}(s' | s, a) V^*(s', t+1) \right]$$

Quiz!

Computational complexity of value iteration

Initialize value function at last time-step

$$V^*(s, T-1) = \min_a c(s, a)$$

for $t = T-2, \ldots, 0$

Compute value function at time-step t

$$V^*(s, t) = \min_a \left[c(s, a) + \gamma \sum_{s'} \mathcal{T}(s' \mid s, a) V^*(s', t+1) \right]$$

When poll is active respond at **PollEv.com/sc2582**

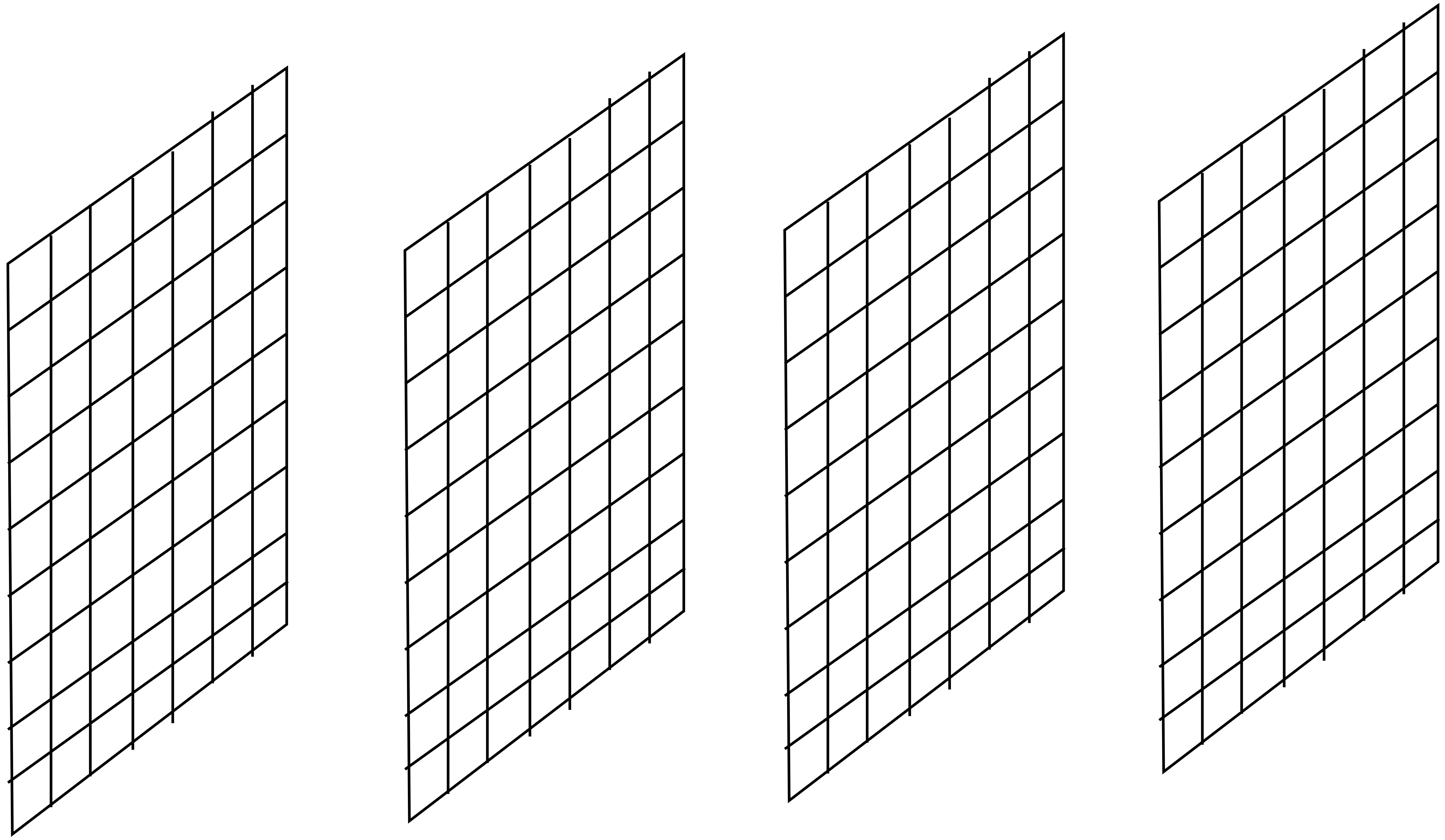
Why is the optimal policy a function of time?

Pulling the goalie
when you
are losing and have
seconds left ..

What happens when horizon is infinity?

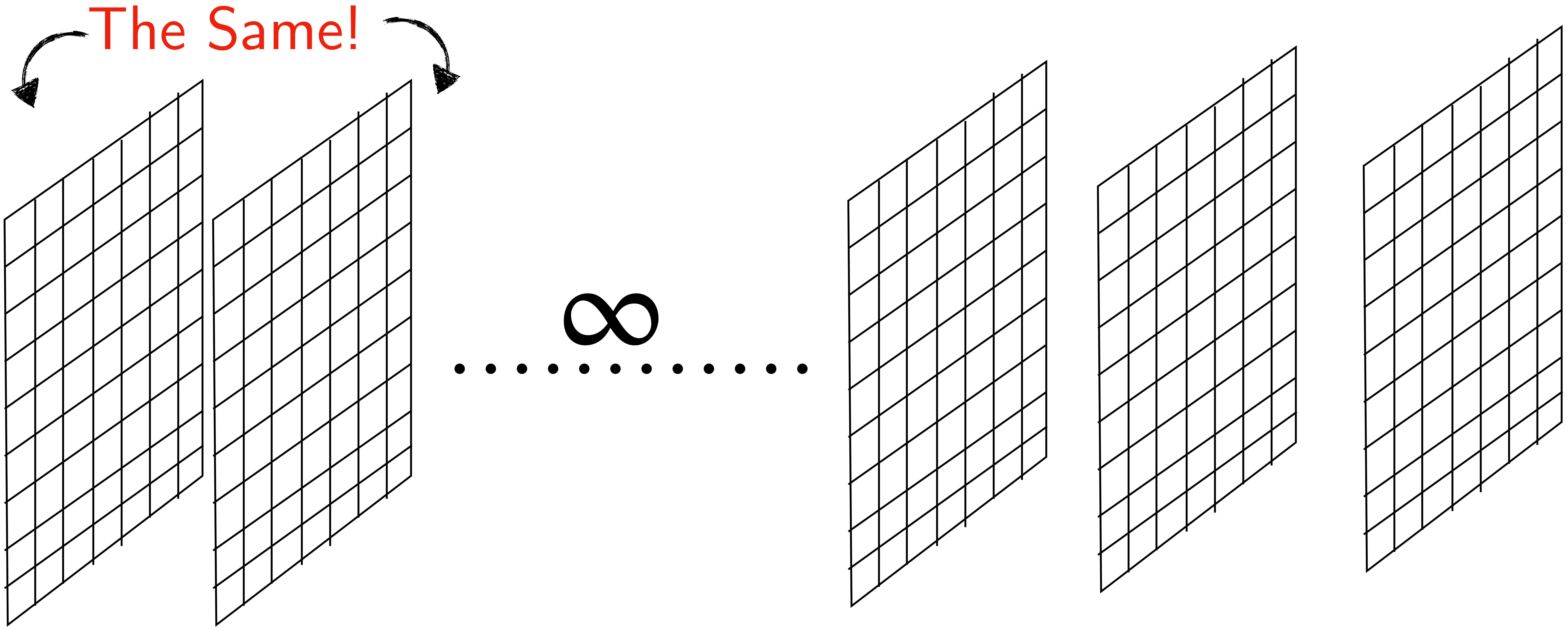
What happens when horizon is infinity?

∞ ………



$$V^{\pi^*}(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} V^{\pi^*}(s_{t+1})) \right]$$

Value Function Converges! (For $\gamma < 1$)



$$V^*(s) = \min_a \left[c(s,a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^*(s)\right]$$

Infinite Horizon Value Iteration

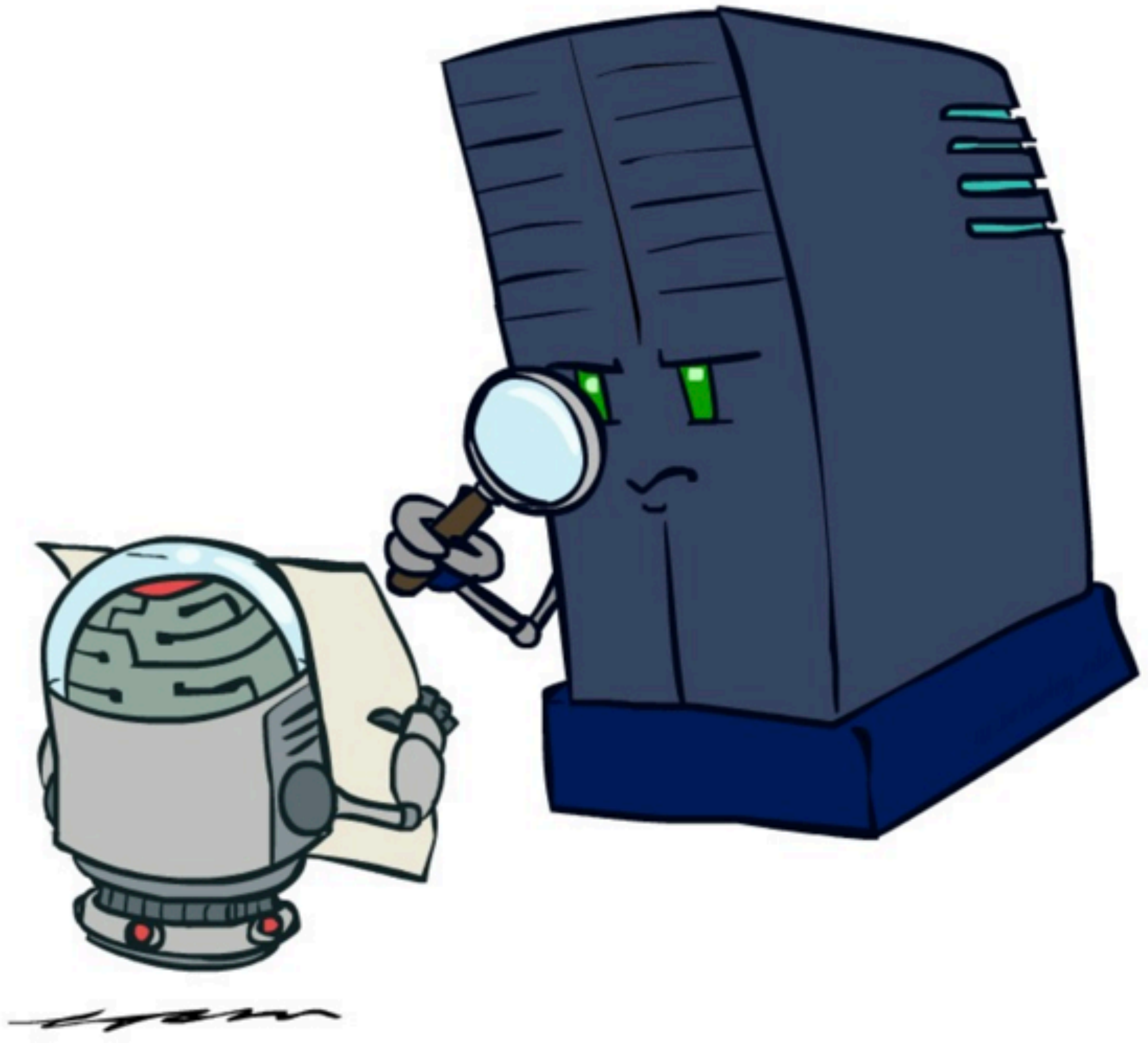
Initialize with some value function $V^*(s)$

Repeat forever

Update values

$$V^*(s) = \min_a \left[c(s,a) + \gamma \sum_{s'} \mathcal{T}(s'|s,a) V^*(s') \right]$$

Policy Iteration



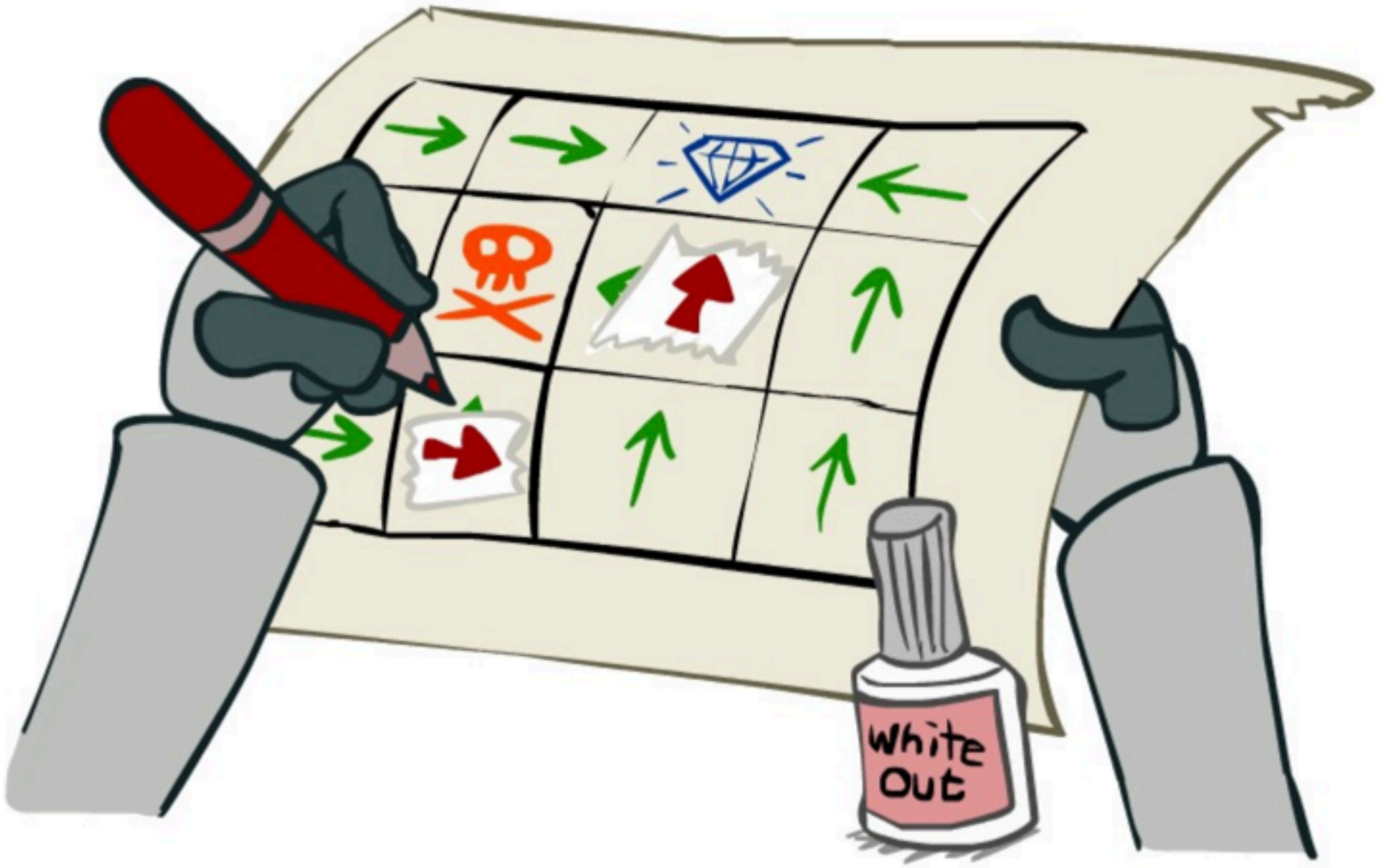
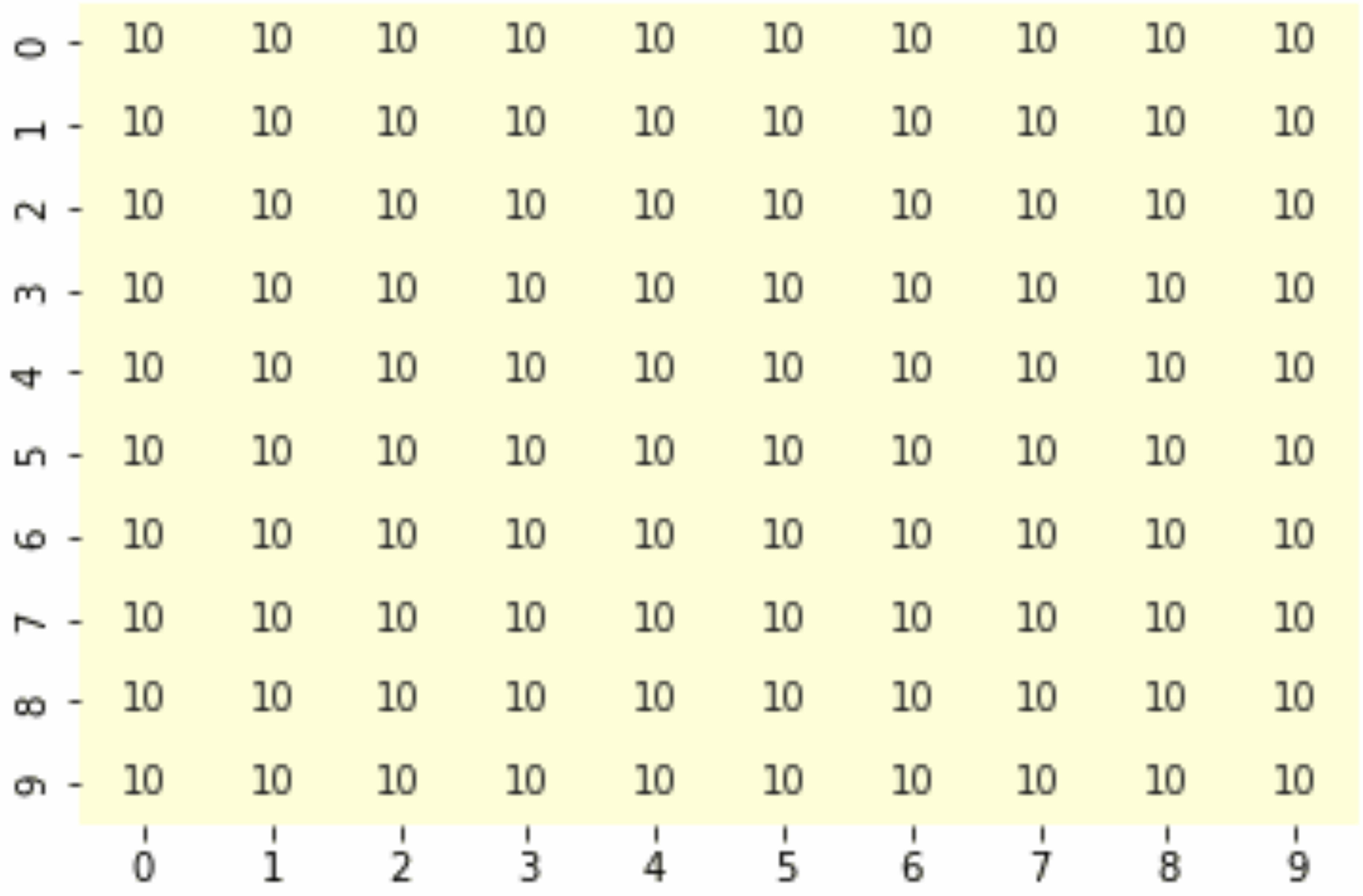
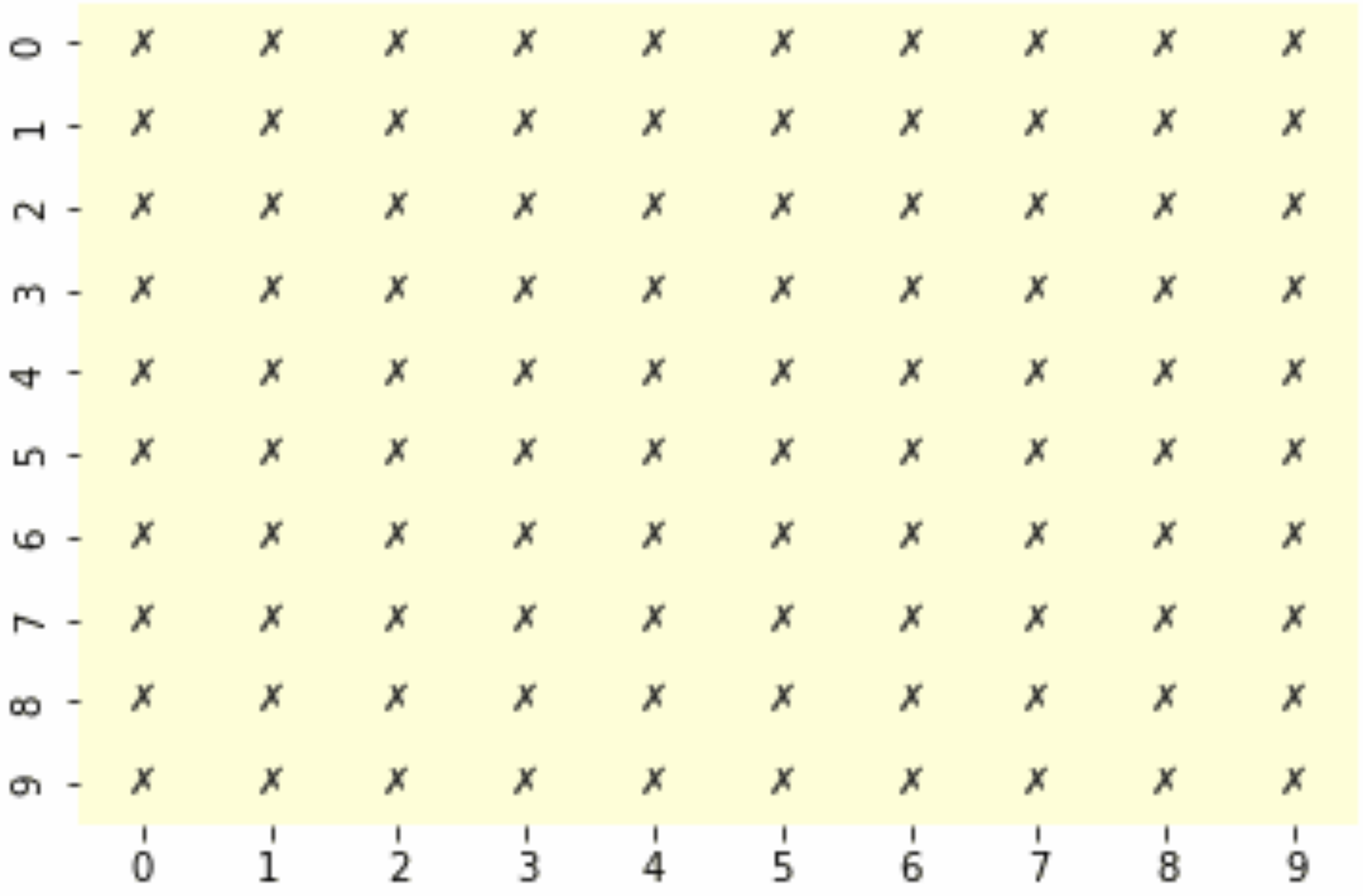


Image courtesy Dan Klein

Which converges faster: value or policy?



Values



Policy

Policy converges faster than the value

Can we iterate over policies?

Policy Iteration

Init with some policy π

Repeat forever

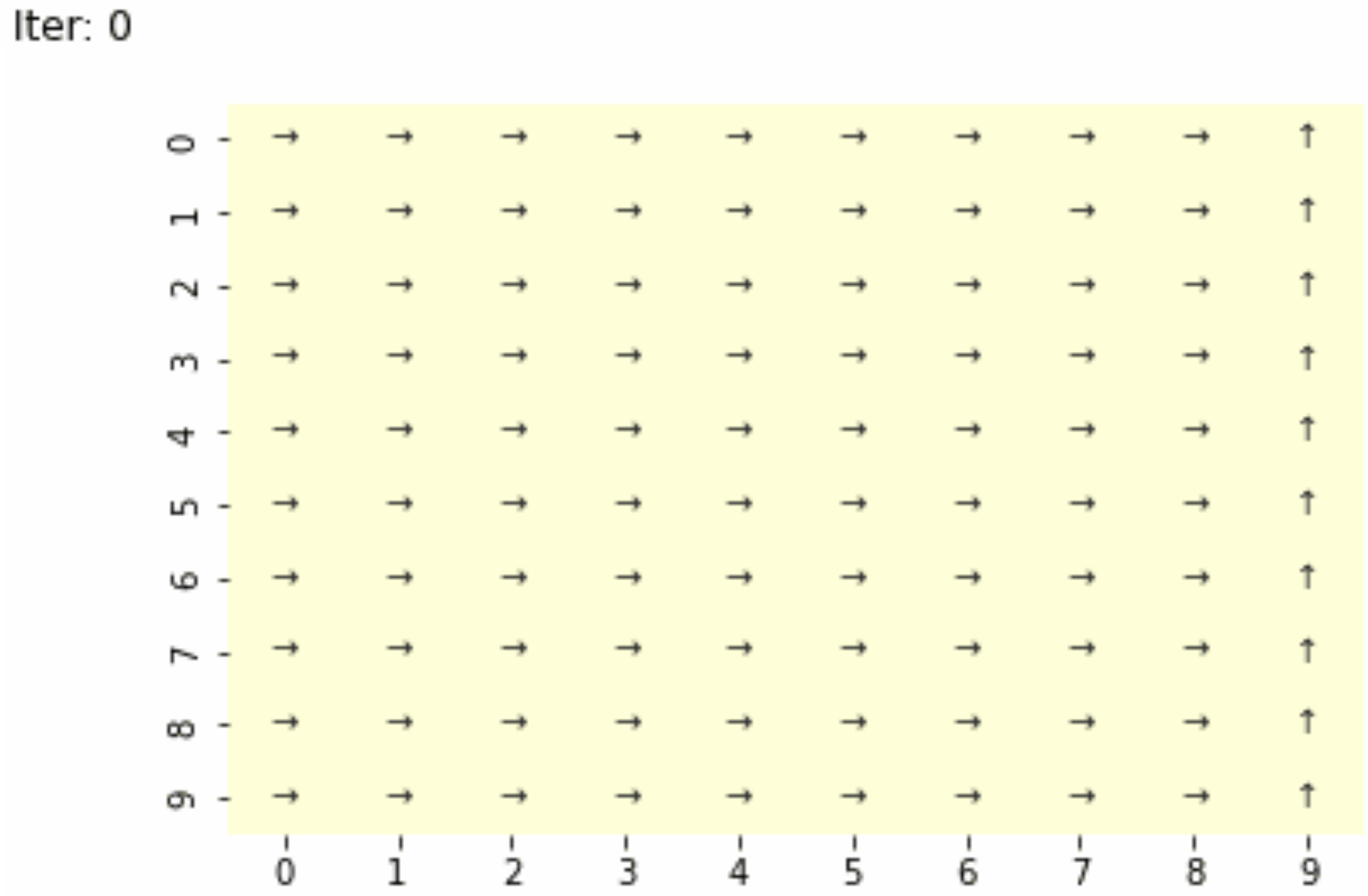
Evaluate policy

$$V^{\pi}(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

Improve policy

$$\pi^{+}(s) = \arg\min_{a} c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

Init with some policy π



Iteration 1

$$V^{\pi}(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

$$\pi^{+}(s) = \arg\min_{a} c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

Policy Iteration

$$V^{\pi}(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

$$\pi^{+}(s) = \arg\min_{a} c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

How do we evaluate policy?

$$V^{\pi}(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{\pi}(s')]$$

Idea 1: Start with an initial guess, and update (like value iteration)

$$V^{i+1}(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^{i}(s')]$$

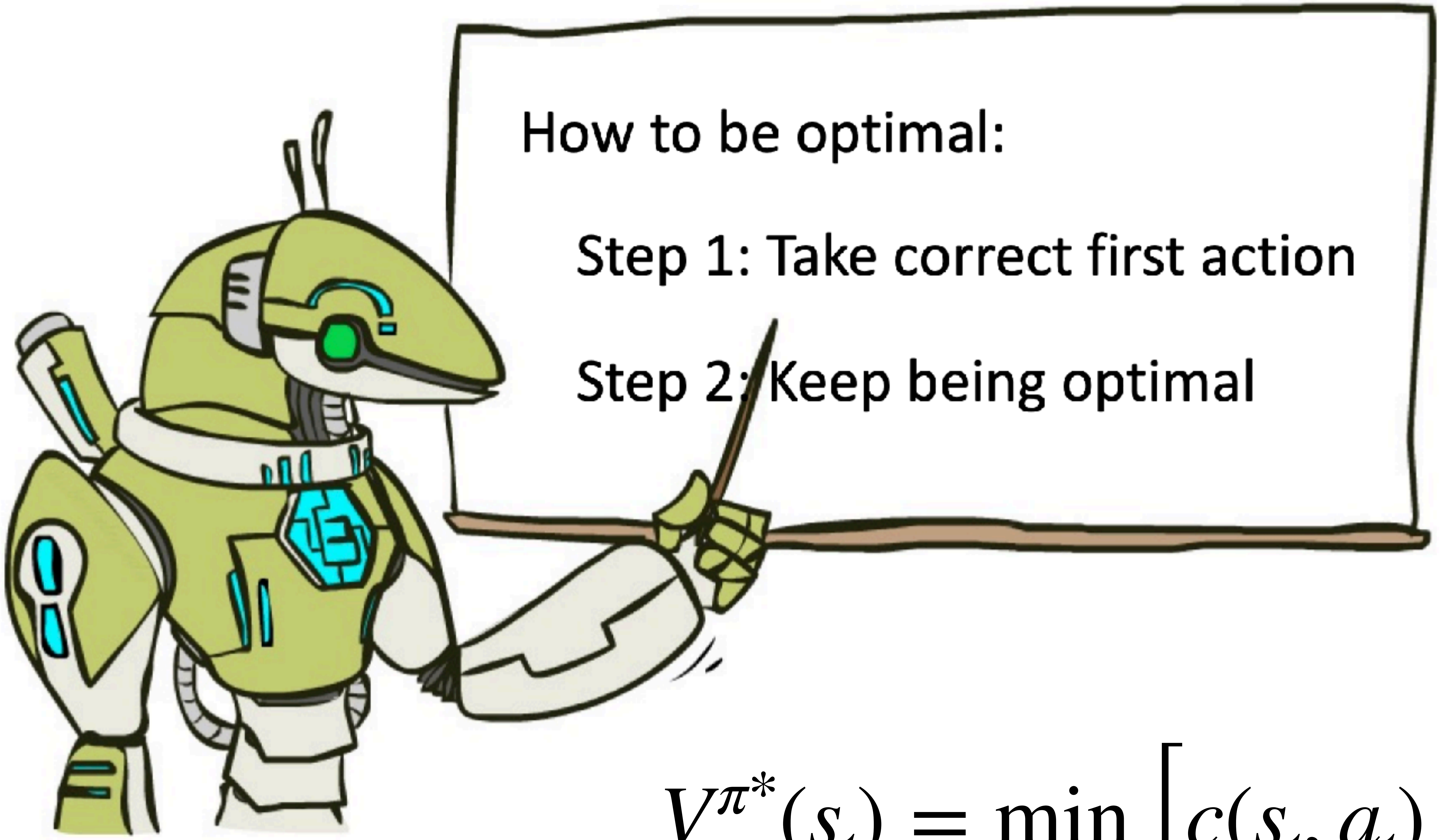
Idea 2: It's a linear set of equations (no max)! Solve for Eigen values

$$\overrightarrow{V^{\pi}} = \overrightarrow{c^{\pi}} + \gamma \mathcal{T}^{\pi} \overrightarrow{V^{\pi}} \longrightarrow \overrightarrow{V^{\pi}} = (1 - \mathcal{T}^{\pi})^{-1} \overrightarrow{c^{\pi}}$$

Value Iteration vs Policy Iteration

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

The Bellman Equation



$$V^{\pi^*}(s_t) = \min_{a_t} \left[c(s_t, a_t) + \gamma \mathbb{E}_{s_{t+1}} V^{\pi^*}(s_{t+1})) \right]$$

Image courtesy Dan Klein

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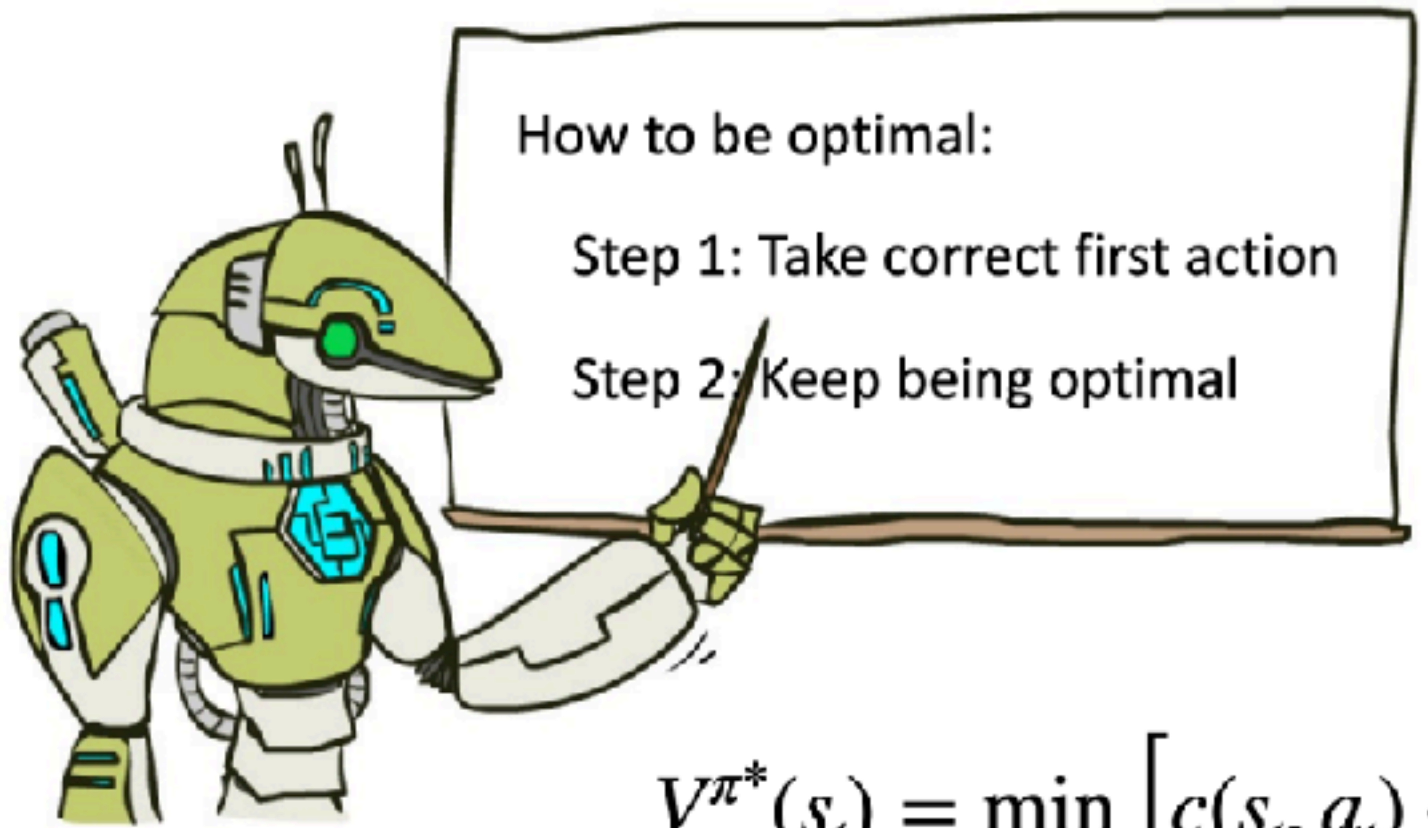
tl;dr

The Bellman Equation



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Policy Iteration

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Repeat forever

Evaluate policy

$$V^\pi(s) = c(s, \pi(s)) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^\pi(s')]$$

Improve policy

$$\pi^+(s) = \arg\min_a c(s, a) + \gamma \mathbb{E}_{s' \sim \mathcal{T}(s,a)} V^\pi(s')]$$